Holomorphic Curves in Moduli Spaces Are Quasi-Isometrically Immersed

Yibo Zhang *

Institut Fourier, UMR 5582, Laboratoire de Mathématiques

Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France

email: yibo.zhang@univ-grenoble-alpes.fr

Abstract

A *holomorphic curve* in moduli spaces is the image of a non-constant holomorphic map from a hyperbolic surface B of type (g, n) to the moduli space $\mathcal{M}_h$ of closed Riemann surfaces of genus h. We show that, when all peripheral monodromies are of infinite order, the holomorphic map is a quasi-isometric immersion with parameters depending only on g, n, h and the systole of B. When peripheral monodromies also satisfy an additional condition, we find a lift quasi-isometrically embedding a fundamental polygon of the hyperbolic surface B into the Teichmüller space. We further improve the Parshin-Arakelov finiteness theorem, by proving that there are only finitely many monodromy homomorphisms induced by holomorphic curves of type (g, n) in $\mathcal{M}_h$ where systole is bounded away from 0, up to equivalence.

Contents

1 Introduction

Let $\mathcal{T}_h$ be the Teichmüller space parameterizing complex structures on a closed oriented smooth surface Σ_h of genus $h \geq 2$, up to isotopy. This space has a complex structure which is induced by an embedding $\mathcal{T}_h \hookrightarrow \mathbb{C}^{3h-3}$, due to Bers and Maskit (see [Ber70] and [Mas70]). The Teichmüller distance on $\mathcal{T}_h$ will be denoted by $d_{\mathcal{T}}$ in the sequel (see Subsection 2.1 for the definition).

The mapping class group Mod_h consists of all orientation-preserving diffeomorphisms of Σ_h up to isotopy. This group acts properly discontinuously on $\mathcal{T}_h$ and the quotient space is the moduli

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space $\mathcal{M}_h$. Each mapping class in Mod_h is a holomorphic automorphism of $\mathcal{T}_h$ and an isometry for the Teichmüller distance.

A *holomorphic disc* in $\mathcal{T}_h$ is the image of a holomorphic map $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ from the upper half plane $\mathbb{H}^2 \subset \mathbb{C}$ to the Teichmüller space. The hyperbolic plane is endowed with the usual complex structure making it biholomorphic to the open unit disc. Passing to the quotient we obtain a map $\mathring{F} : \mathbb{H}^2 \to \mathcal{M}_h$ to the moduli space. Let $\Gamma \leq \mathrm{Stab}(\mathring{F}) := \{\phi \in \mathrm{Aut}(\mathbb{H}^2) \mid \mathring{F} \circ \phi = \mathring{F}\}$ be a lattice and set $B := \Gamma \backslash \mathbb{H}^2$. Suppose that B is an oriented surface of genus g with n cusps (without boundary). The image of the quotient map $F : B \to \mathcal{M}_h$ is then a *holomorphic curve* of type (g, n) in $\mathcal{M}_h$.

We use d_B to denote the hyperbolic distance on B. The *systole* of B, denoted by $\mathrm{sys}(B)$, is the length of the shortest essential (i.e. non-contractible and non-peripheral) closed curve. A *cusp region* of B, usually denoted by U, is the neighbourhood of a cusp bounded by a horocycle of length 2.

As in [IS88, p.212] and [Moi77, p.176], a holomorphic map $F : B \to \mathcal{M}_h$ induces a group homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$ which is called a *monodromy homomorphism* of F (see Subsection 2.2 for the definition). The image $F_*([\gamma])$ along a based loop $\gamma \subset B$ that goes once or several times around a cusp, clockwise or counterclockwise, is called a *peripheral monodromy* of the cusp.

As B and $\mathcal{T}_h$ are complex manifolds, they are endowed with intrinsic *Kobayashi pseudo-norms*, which need not be positive definite, as follows.

Definition 1.1. Let X be a complex manifold. The *Kobayashi pseudo-norm* $\mathrm{Kob}_X : TX \to \mathbb{R}_{\geq 0}$ on X is defined by $\mathrm{Kob}_X(x, v) = \inf_\phi\{1/c\}$ for $x \in X$ and $v \in T_xX$, where the infimum is taken over all holomorphic maps ϕ from the unit disc in $\mathbb{C}$ to X satisfying $\phi(0) = x$ and $(d\phi)_0(\partial/\partial z) = c \cdot v$.

A smooth manifold X with a pseudo-norm $TX \ni (x, v) \mapsto K(x, v)$ is also endowed with a pseudo-distance $d_{X,K} : X \times X \to \mathbb{R}$ given by the formula

$$d_{X,K}(x_1, x_2) = \inf_\gamma \int_0^1 K(\gamma(t), \dot{\gamma}(t))dt$$

where the infimum is taken over all piecewise smooth paths joining x_1 to x_2. The pseudo-distance $d_{X,\mathrm{Kob}}$ induced by Kob_X is called the *Kobayashi distance* on X. If Kob_X is a norm, i.e. it vanishes only at $0 \in TX$, then X is said to be *Kobayashi hyperbolic.*

Both $\mathbb{H}^2$ and $\mathcal{T}_h$ are Kobayashi hyperbolic. The Kobayashi (pseudo-)norm on $\mathbb{H}^2$ coincides with the norm of the Poincaré Riemannian metric or half the hyperbolic metric. The Kobayashi distance on $\mathbb{H}^2$ coincides with half the hyperbolic distance, i.e.

$$\mathrm{Kob}_{\mathbb{H}^2}(z, v) = \frac{1}{2}\frac{|dz(v)|}{\mathrm{Im}(z)} \quad \text{and} \quad d_{\mathbb{H}^2,\mathrm{Kob}}(z_1, z_2) = \tanh^{-1}\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|} = \frac{1}{2}d_{\mathbb{H}^2}(z_1, z_2)$$

for $z, z_1, z_2 \in \mathbb{H}^2$ and $v \in T_z\mathbb{H}^2$ (see [Aba89, Proposition 2.3.4]). On the other hand, the Kobayashi (pseudo-)norm $\mathrm{Kob}_\mathcal{T}$ on $\mathcal{T}_h$ is a Finsler metric and the Kobayashi distance on $\mathcal{T}_h$ coincides with the Teichmüller distance (see [Roy71, Theorem 3]), i.e. $d_{\mathcal{T},\mathrm{Kob}} = d_\mathcal{T}$.

Definition 1.2. Let $F : B \to \mathcal{M}_h$ be a holomorphic map whose lift is $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$.

- The distance $d_{\widetilde{F}}$ on $\mathbb{H}^2$ is induced by the pullback pseudo-norm $\widetilde{F}^* \mathrm{Kob}_\mathcal{T}$.
- The distance d_F on B is defined by

$$d_F(b_1, b_2) = \inf\{d_{\widetilde{F}}(\widetilde{b_1}, \widetilde{b_2}) \mid \widetilde{b_1} \in \mathbb{H}^2 \text{ is a lift of } b_1, \widetilde{b_2} \in \mathbb{H}^2 \text{ is a lift of } b_2\}.$$

- The *Teichmüller distance* $d_\mathcal{M}$ on $\mathcal{M}_h$ is defined by

$$d_\mathcal{M}(q_1, q_2) = \inf\{d_\mathcal{T}(\widetilde{q_1}, \widetilde{q_2}) \mid \widetilde{q_1} \in \mathcal{T}_h \text{ is a lift of } q_1, \widetilde{q_2} \in \mathcal{T}_h \text{ is a lift of } q_2\}.$$

To discuss the rigidity of $F : B \to \mathcal{M}_h$ and $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$, we recall the basic definition of a quasi-isometric embedding and define an extra notion of rigidity for $F : B \to \mathcal{M}_h$.

Definition 1.3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Given $\lambda \geq 1$ and $\epsilon \geq 0$, a map $f : X_1 \to X_2$ is called a (λ, ϵ)-*quasi-isometric embedding* if

$$d_1(x, y)/\lambda - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon$$

for all $x, y \in X_1$. In particular, when f is a $(1, 0)$-quasi-isometric embedding, we say that f is an *isometric embedding.*

Definition 1.4. Given $\lambda \geq 1$ and $\epsilon \geq 0$, a holomorphic map $F : B \to \mathcal{M}_h$ is called a (λ, ϵ)-*quasi-isometric immersion* if

$$(1/2)d_B(b_1, b_2)/\lambda - \epsilon \leq d_F(b_1, b_2) \leq (\lambda/2)d_B(b_1, b_2) + \epsilon$$

for all $b_1, b_2 \in B$. In this case, we say that $F(B)$ is *quasi-isometrically immersed.* In particular, when F is a $(1, 0)$-quasi-isometric immersion, we say that F is an *isometric immersion* and $F(B)$ is *isometrically immersed.*

Just as a quasi-isometric embedding needs not be an embedding, a quasi-isometric immersion needs not be an immersion in the usual sense. However, an isometric immersion is indeed an immersion.

A holomorphic map $f : X_1 \to X_2$ between complex manifolds is distance-decreasing for the intrinsic Kobayashi distances, namely $d_{X_1,\mathrm{Kob}}(x, y) \geq d_{X_2,\mathrm{Kob}}(f(x), f(y))$ (see, e.g., [Aba89, Proposition 2.3.1]). In particular, the holomorphic map $F : B \to \mathcal{M}_h$ and its lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ satisfy the following inequalities:

$$\frac{1}{2}d_B\Big(b_1, b_2\Big) \geq d_F\Big(b_1, b_2\Big) \geq d_{\mathcal{M}}\Big(F(b_1), F(b_2)\Big),$$
$$\frac{1}{2}d_{\mathbb{H}^2}\Big(\widetilde{b_1}, \widetilde{b_2}\Big) \geq d_{\widetilde{F}}\Big(\widetilde{b_1}, \widetilde{b_2}\Big) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{b_1}), \widetilde{F}(\widetilde{b_2})\Big).$$

Consequently, each peripheral monodromy must be either reducible or of finite order (see Corollary 2.4). In particular, if a peripheral monodromy ϕ is of infinite order, then a power ϕ^{μ} is a multi-twist.

An isometrically immersed curve $F(B) \subset \mathcal{M}_h$ is known as a *Teichmüller curve.* The first non-trivial cases were discovered by Veech ([Vee89]) and all Teichmüller curves in $\mathcal{M}_2$ and $\mathcal{M}_3$ are almost well-understood ([McM05; McM06] and [LN14; McM23, Theorem 5.5]). In $\mathcal{M}_h$ with $h \geq 5$, Teichmüller curves are elusive and the only known primitive case was given in [BM10].

Quasi-isometric rigidity for holomorphic curves. Every isometric immersion from B to $\mathcal{M}_h$ is either holomorphic or anti-holomorphic (see [Ant17]). However, a holomorphic map needs not be an isometric immersion. The following theorem shows that a weaker statement still holds.

Theorem A. *Given (g, n), h and ϵ with $2g - 2 + n > 0$, $h \geq 2$ and $\epsilon > 0$, there exists a constant $K = K(g, n, h, \epsilon)$ that depends only on g, n, h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) with $\mathrm{sys}(B) \geq \epsilon$ and cusp regions $U_1, \ldots, U_n \subset B$. Let $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$.*

- *(i) For each $i = 1, \ldots, n$, if a peripheral monodromy of the i-th cusp is of infinite order, then $F|_{U_i} : (U_i, (1/2)d_B) \to (\mathcal{M}_h, d_{\mathcal{M}})$ is a $(1, K)$-quasi-isometric embedding.*
- *(ii) If all peripheral monodromies are of infinite order, then F is a $(1, K)$-quasi-isometric immersion.*

Given a monodromy homomorphism F_*, if a peripheral monodromy of a fixed cusp is of infinite order, then all peripheral monodromies of this cusp are of infinite order. Therefore, the hypothesis in Theorem A - (ii) can be checked for only one peripheral monodromy of each cusp.

This result is optimal. On the one hand, there exists a holomorphic curve in $\mathcal{M}_h$ that is quasi-isometrically but not isometrically immersed, see Example 5.2. On the other hand, if peripheral monodromies of the i-th cusp are of finite order, then $F|_{U_i} : (U_i, (1/2)d_B) \to (\mathcal{M}_h, d_{\mathcal{M}})$ needs not be a quasi-isometric embedding. Example 5.5 shows such a possibility when $F(U_i)$ is located in the thick part of $\mathcal{M}_h$.

Quasi-isometric rigidity for fundamental domains. If the holomorphic map $F : B \to \mathcal{M}_h$ is an isometric immersion, then the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ is a complex geodesic for the intrinsic Kobayashi norms. Teichmüller's uniqueness theorem (see, e.g., [FM11, Theorems 11.8 and 11.9]) shows that any two points of the Teichmüller space $\mathcal{T}_h$ are joined by a unique real geodesic. Therefore, the lift $\widetilde{F} : (\mathbb{H}^2, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ is an isometric embedding.

If the holomorphic map $F : B \to \mathcal{M}_h$ is a quasi-isometric immersion, in general, the lift $\widetilde{F} : (\mathbb{H}^2, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ fails to be a quasi-isometric embedding.

We now aim to obtain a hyperbolic polygon $D \subset \mathbb{H}^2$, i.e. a fundamental domain of B bounded by geodesic segments, such that $\widetilde{F}\Big|_D : (D, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ is a quasi-isometric embedding. Given a holomorphic map $F : B \to \mathcal{M}_h$ and a monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$, we start with a suitable condition on the monodromy.

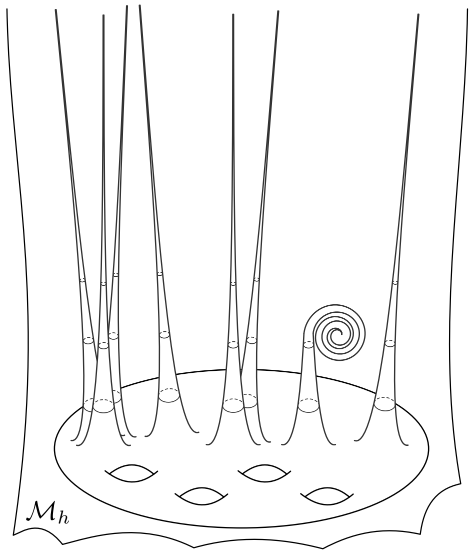

Figure 1: A cartoon of a holomorphic curve in $\mathcal{M}_h$. Each cusp neighbourhood of the holomorphic curve almost keeps the hyperbolic structure along the Teichmüller metric, unless the peripheral monodromy is of finite order.

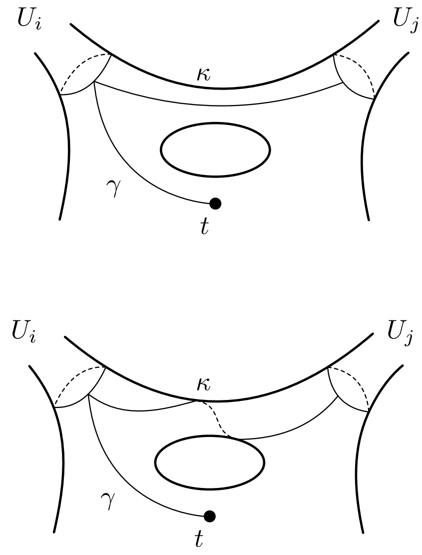

Figure 2: Different geodesic segments joining the boundaries of two given cusp regions U_i and U_j, each providing a pair of peripheral monodromies (ϕ_i, ϕ_j).

Definition 1.5 (*disjointed* mapping classes)**.** Let ϕ_1 and $\phi_2 \in \mathrm{Mod}_h$ be reducible mapping classes. Suppose that there exist positive integers μ_1, μ_2 and multi-curves $\boldsymbol{\alpha_1} = \{\alpha_{1,1}, \ldots, \alpha_{1,m_1}\}$, $\boldsymbol{\alpha_2} = \{\alpha_{2,1}, \ldots, \alpha_{2,m_2}\}$ such that $\phi_1^{\mu_1}$, $\phi_2^{\mu_2}$ are multi-twists along $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$,

$$\phi_1^{\mu_1} = T_{\alpha_{1,1}}^{r_{1,1}} \circ \cdots \circ T_{\alpha_{1,m_1}}^{r_{1,m_1}}, \qquad \phi_2^{\mu_2} = T_{\alpha_{2,1}}^{r_{2,1}} \circ \cdots \circ T_{\alpha_{2,m_1}}^{r_{2,m_1}}$$

with $r_{i,j} \in \mathbb{Z} \setminus \{0\}$, for $i = 1, 2$ and $j = 1, \ldots, m_i$. We say that ϕ_1 and ϕ_2 are *disjointed* if pairs of curves in $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ are disjoint or coincide.

Definition 1.6 (*disjointed* peripheral monodromies)**.** Let U_i, U_j be cusp regions of the hyperbolic surface B, $i \neq j$, endowed with a geodesic segment κ joining ∂U_i to ∂U_j. Set $\{t_0\} = \partial U_i \cap \kappa$. and take an arbitrary path γ joining t to t_0 (see Figure 2). The loop along $\gamma \cup \partial U_i$ based at t that goes once around U_i clockwise is denoted by γ_i and its monodromy is denoted by ϕ_i. The loop along $\gamma \cup \kappa \cup \partial U_j$ based at t that goes once around U_j clockwise is denoted by γ_j and its monodromy is denoted by ϕ_j. We say that peripheral monodromies of U_i and U_j are *disjointed* along κ if ϕ_i and ϕ_j are reducible and disjointed.

A pair of disjointed mapping classes after a simultaneous conjugacy is again disjointed. Therefore, a pair of peripheral monodromies being disjointed along κ is independent of the choice of the path γ. However, a different choice of κ changes the peripheral monodromies by a non-simultaneous conjugacy.

Theorem B. *Given (g, n), h and ϵ with $2g - 2 + n > 0$, $h \geq 2$ and $\epsilon > 0$, there exists a constant $K = K(g, n, h, \epsilon)$ that depends only on g, n, h, ϵ and satisfies the following statement. Let $B = \Gamma \backslash \mathbb{H}^2$ be an oriented hyperbolic surface of type (g, n) with $\mathrm{sys}(B) \geq \epsilon$ and cusp regions $U_1, \ldots, U_n \subset B$. Let $D \subset \mathbb{H}^2$ be a fundamental convex polygon of B with exactly n ideal points. Let $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$. If*

- *peripheral monodromies of all cusps are of infinite order,*
- *for each $i \neq j$, $i = 1, \ldots, n$ and $j = 1, \ldots, n$, there exists a geodesic segment $\kappa_{i,j} \subset B$ joining ∂U_i to ∂U_j with a lift $\widetilde{\kappa_{i,j}} \subset D$ such that peripheral monodromies of U_i and U_j are not disjointed along $\kappa_{i,j}$,*

then $\widetilde{F}\Big|_D : (D, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ *is a $(2, K + \mathrm{diam}(D))$-quasi-isometric embedding.*

When $(g, n) = (0, n)$, one can describe the monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$ as an n-tuple in Mod_h as follows. Choose homotopy classes of loops $\gamma_1, \ldots, \gamma_n \subset B$ based at t such that γ_i goes around the i-th cusp exactly once clockwise and $\gamma_n \cdots \gamma_1$ is homotopically trivial, for $i = 1, \ldots, n$. The n-tuple $(\phi_1, \ldots, \phi_n) \coloneqq (F_*(\gamma_1), \ldots, F_*(\gamma_n))$ is called a *global monodromy* of F. Note that $\phi_1 \cdots \phi_n = 1$.

On the one hand, there exists a fundamental polygon $D \subset \mathbb{H}^2$ of B and segments $\kappa_{i,j} \subset B$ together with lifts $\widetilde{\kappa_{i,j}} \subset D$ that satisfy the following equivalence: the peripheral monodromies of U_i and U_j are disjointed along $\kappa_{i,j}$ if and only if ϕ_i and ϕ_j are disjointed. On the other hand, a different choice of $\gamma_1, \ldots, \gamma_n$ changes $(\phi_1, \ldots, \phi_n)$ by a sequence of Hurwitz moves:

$$(\ldots, \phi_i \circ \phi_{i+1} \circ \phi_i^{-1}, \phi_i, \ldots) \xleftarrow{L_i} (\ldots, \phi_i, \phi_{i+1}, \ldots) \xrightarrow{R_i} (\ldots, \phi_{i+1}, \phi_{i+1}^{-1} \circ \phi_i \circ \phi_{i+1}, \ldots).$$

Thereby, the existence of a fundamental polygon D as in Theorem B such that $\widetilde{F}\Big|_D$ is a quasi-isometric embedding is equivalent to the existence of a tuple being Hurwitz equivalent to $(\phi_1, \ldots, \phi_n)$ whose components are of infinite order and pairwise non-disjointed.

The second hypothesis of Theorem B sometimes is a mild condition. In particular, a holomorphic curve of type $(0, n), n \geq 3$ in $\mathcal{M}_2$ with each peripheral monodromy the Dehn twist along a non-separating closed curve (i.e. the image of the classifying map of a genus-2 Lefschetz fibration without reducible fibres) must have a quasi-isometrically embedded fundamental polygon (see Subsection 5.3).

Finiteness result for holomorphic curves. Let C be a 2-dimensional complex manifold and B be a Riemann surface of type (g, n). Consider a holomorphic projection $\pi : C \to B$ such that the generic fibres are compact Riemann surfaces of genus h and the branch locus is a finite subset $S \subset B$. Such a holomorphic map is called a genus-h *holomorphic fibration* over B, denoted by C/B. Set $C_b = \pi^{-1}(b)$ for $b \in B$. When $S = \emptyset$, the holomorphic fibration is further called a genus-h *holomorphic family*.

Therefore, given a holomorphic fibration C/B, then $\big(C \setminus \pi^{-1}(S)\big)/\big(B \setminus S\big)$ is a holomorphic family over a Riemann surface of type $(g, n + |S|)$.

A holomorphic family C/B is *isotrivial* if the fibres are all biholomorphic, i.e. $C_{b_1} \cong C_{b_2}$ for $b_1, b_2 \in B$. Two holomorphic families C/B and C'/B over B are *isomorphic* if there exists a biholomorphic map between C and C' that preserves fibrations.

A holomorphic family C/B always determines the unique holomorphic map $F : B \to \mathcal{M}_h$, which is called the *classifying map* and denoted by $\mathbf{CM}(C/B) = F$. Then a monodromy homomorphism of C/B in the usual sense is a monodromy homomorphism of its classifying map.

Conversely, there exists a universal holomorphic family over $\mathcal{T}_h$ such that the fibre at $q \in \mathcal{M}_h$ is biholomorphic to the Riemann surface parameterized by q (see [EF85, Theorem 1] and [Nag88, p.349]). The pullback of the universal family by the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ of a holomorphic map $F : B \to \mathcal{M}_h$ induces a holomorphic family C/B. A holomorphic map could be the classifying map of at most

$$\# \operatorname{Hom}(\pi_1(B, t), \operatorname{Aut}(C_t)) \leq (2g + n)^{84(h-1)}$$

many non-isotrivial non-isomorphic holomorphic families.

The geometric Shafarevich conjecture, now known as the Parshin-Arakelov finiteness (see [Par68; Ara72; IS88]), claims that there are only finitely many non-isotrivial non-isomorphic holomorphic families of Riemann surfaces of genus h over a given base space B (assuming $h \geq 2$ and B hyperbolic).

The finiteness of holomorphic families corresponds to the finiteness of monodromy homomorphisms up to conjugacy. On the one hand, a class of homomorphisms in

$$M_h(B, t) \coloneqq \operatorname{Hom}(\pi_1(B, t), \mathrm{Mod}_h) / \mathrm{Mod}_h$$

determines the topology of C/B. On the other hand, non-isotrivial holomorphic families over B having the same class of monodromy homomorphism in $M_h(B, t)$ are isomorphic (see Rigidity Theorem in [IS88]).

Our next goal is to analyse holomorphic families over homeomorphic but non-biholomorphic Riemann surfaces and compare their monodromy homomorphisms.

Take an oriented smooth surface $\Sigma_{g,n}$ of type (g, n) and fix $s \in \Sigma_{g,n}$. Given a holomorphic map $F : B \to \mathcal{M}_h$ and a monodromy homomorphism $F_* \in \operatorname{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$, there exists an orientation preserving diffeomorphism $f_B : \Sigma_{g,n} \to B$ such that $f_B(s) = t$. Therefore, f_B pulls F_*

back to $(f_B)^*(F_*) = F_* \circ (f_B)_* \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}, s), \mathrm{Mod}_h)$. A different choice of f_B, however, does not change the corresponding class in

$$M_{g,n,h} := \mathrm{Mod}_{g,n} \setminus \mathrm{Hom}(\pi_1(\Sigma_{g,n}, s), \mathrm{Mod}_h) / \mathrm{Mod}_h\,. \tag{1}$$

Thus to a holomorphic map $F : B \to \mathcal{M}_h$ one associates the class $\mathbf{MO}(F) := [F_* \circ (f_B)_*] \in M_{g,n,h}$.

Two holomorphic fibrations, even if non-holomorphic, correspond to the same class in $M_{g,n,h}$ if and only if they are isomorphic after removing singular fibres. The set $M_{g,n,h}$ is infinite. Moreover, at least for certain g, n and $h \geq 3$, there exist infinitely many symplectic Lefschetz fibrations with pairwise non-homeomorphic total spaces (see [FS04]). Therefore, the subset of classes realised by symplectic Lefschetz fibrations is also infinite. However in [Cap02] Caporaso proved that there is a uniform, i.e. independent of B, bound for the number of classes in $M_{g,n,h}$, that can be realised by a genus-h holomorphic fibration over a Riemann surface B of type $(g, 0)$ having n branch points, see also [Hei04; Del16]. The following theorem is an algebraic improvement of the Parshin-Arakelov finiteness, which improves Corollary 2 in [Shi97].

Theorem C. *Given (g, n), h and ϵ with $2g - 2 + n > 0$, $h \geq 2$ and $\epsilon > 0$, then the subset*

$$\left\{ \boldsymbol{MO}\big(\,\boldsymbol{CM}(C/B)\big) \,\middle|\, \begin{array}{l} B \textit{ is a Riemann surface of type } (g,n) \textit{ with } \mathrm{sys}(B) \geq \epsilon \\ C/B \textit{ is a non-isotrivial genus-}h\textit{ holomorphic family over } B \end{array} \right\} \subset M_{g,n,h}$$

is finite.

The following finiteness result is an immediate consequence of the above theorem.

Corollary 1.7 (Theorem 6.5 in [Shi14])**.** *Given (g, n) and h with $2g - 2 + n > 0$ and $h \geq 2$, there are only finitely many Teichmüller curves of type (g, n) in $\mathcal{M}_h$.*

Two holomorphic curves $F_1 : B_1 \to \mathcal{M}_h$ and $F_2 : B_2 \to \mathcal{M}_h$ are called *homotopic* if B_1, B_2 are of the same type (g, n) and there exist orientation preserving diffeomorphisms $f_1 : \Sigma_{g,n} \to B_1$, $f_2 : \Sigma_{g,n} \to B_2$ such that $F_1 \circ f_1$, $F_2 \circ f_2$ are homotopic. Theorem C implies that there are only finitely many holomorphic curves of type (g, n) in $\mathcal{M}_h$ up to homotopy when hyperbolic systoles are bounded away from 0.

Outline. Subsections 2.1 and 2.2 introduces the main notions needed for studying the Teichmüller space and monodromies of a holomorphic map $F : B \to \mathcal{M}_h$. Subsection 2.3 presents some condition on the monodromy, which partially describes a Teichmüller curve and is used to prove Corollary 1.7. Subsection 2.4 revisits Mumford's compactness of thick moduli spaces and provides some tools that we will need later.

In Section 3, using the irreducibility of holomorphic fibrations, we provide the auxiliary Theorem 3.2 which claims that each non-constant holomorphic curve has a non-empty intersection with a certain thick part of the moduli space. The proof of Theorem C also appears in this section.

For our rigidity results, we first investigate a holomorphic hyperbolic cusp region in $\mathcal{M}_h$ in Subsection 4.1. We emphasise that a mapping class of infinite order changes a marked hyperbolic surface slightly in $(\mathcal{T}_h, d_{\mathcal{T}})$ only if the mapping class is a multi-twist along small closed geodesics on the hyperbolic surface. This fact implies that a holomorphic hyperbolic cusp region in $\mathcal{M}_h$ is affected by a "force" from the cusp, which partially shows the quasi-isometric embedding in Theorem A - (i). Theorem 3.2 pulls the holomorphic curve as well as each holomorphic hyperbolic cusp region using another "force" from the thick part of the moduli space, hence we prove Theorem A in Subsection 4.2. The proof of Theorem B appears in Subsection 4.3. Finally, Section 5 provides some examples and an application on genus-2 Lefschetz fibrations.

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2 Preliminary

We fix non-negative integers g, n and h with $2g - 2 + n > 0$ and $h \geq 2$ once and for all.

2.1 Teichmüller space. Let $\Sigma_{g,n}$ be an oriented smooth surface of genus g with n punctures without boundary. The *geometric intersection number* of two closed curves γ_1 and γ_2, denoted by $\iota(\gamma_1, \gamma_2)$, is the minimum cardinality of $\nu_1 \cap \nu_2$ for all closed curves ν_1, ν_2 such that γ_i is homotopic to ν_i, for $i = 1, 2$.

We say that a set of closed curves $\{\gamma_1, \ldots, \gamma_k\}$ *fills up* the surface $\Sigma_{g,n}$ if, for any non-contractible non-peripherial closed curve ν, $\iota(\nu, \gamma_i) \geq 1$ for some i. A set of disjoint simple closed curves on $\Sigma_{g,n}$ is called a multi-curve. By convention, we define the product $\gamma_1 \cdot \gamma_2$ of two oriented paths as their concatenation and the inverse γ^{-1} of an oriented path is the same path with the opposite orientation.

The Teichmüller space $\mathcal{T}_{g,n}$ consists of all marked Riemann surfaces of type (g, n), i.e. equivalent pairs (X, f_X) where X is a Riemann surface and $f_X : \Sigma_{g,n} \to X$ is an orientation preserving diffeomorphism. Two pairs (X, f_X), (Y, f_Y) are equivalent if $f_Y \circ f_X^{-1} : X \to Y$ is isotopic to a biholomorphism. Based on the uniformization of Riemann surfaces, each equivalent class is represented by a marked hyperbolic surface. Then a mapping class $[\phi] \in \mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by $[\phi] \cdot [(X, f_X)] = [(X, f_X \circ \phi^{-1})]$. The mapping class group acts properly discontinuously on the Teichmüller space and the quotient space is called the moduli space, denoted by $\mathcal{M}_{g,n}$.

For any two points $[(X, f_X)]$, $[(Y, f_Y)] \in \mathcal{T}_{g,n}$, we define the *Teichmüller distance* $d_{\mathcal{T}}$ by

$$d_{\mathcal{T}}([(X, f_X)], [(Y, f_Y)]) = \frac{1}{2} \log \inf_{\phi} \{K(\phi)\}. \tag{2}$$

Here, the infimum is taken over all quasiconformal diffeomorphisms $\phi : X \to Y$ homotopic to $f_Y \circ f_X^{-1}$, i.e. all quasiconformal diffeomorphisms respecting the markings. Moreover, $K(\phi) \geq 1$ denotes dilatation of ϕ.

The *geodesic length function* $L_\gamma([(X, f_X)])$ assigns to each closed curve $\gamma \subset \Sigma_{g,n}$ the length of the unique geodesic homotopic to $f_X(\gamma)$ on the hyperbolic representative X of $[(X, f_X)] \in \mathcal{T}_{g,n}$. The length of a closed curve γ on a hyperbolic surface X is denoted by $l_X(\gamma)$. The next lemma is due to Wolpert.

Lemma 2.1 ([Wol79]). *Consider two points in $\mathcal{T}_{g,n}$ that are represented by the marked hyperbolic surfaces (X, f_X) and (Y, f_Y). Set $K = \exp\big(2d_{\mathcal{T}}([(X, f_X)], [(Y, f_Y)])\big)$. Then, the geodesic length function is distorted by a factor of at most K, i.e.*

$$\frac{1}{K} L_\gamma([(X, f_X)]) \leq L_\gamma([(Y, f_Y)]) \leq K L_\gamma([(X, f_X)])$$

for any closed curve $\gamma \subset \Sigma_{g,n}$.

Recall the classification of mapping classes due to Bers ([Ber78]). Let $\varphi \in \mathrm{Mod}_h$ be a mapping class. The *translation distance* of φ is defined by

$$\tau(\varphi) = \inf_{[(X, f_X)] \in \mathcal{T}_h} d_{\mathcal{T}}([(X, f_X)], \varphi \cdot [(X, f_X)]).$$

Then $\tau(\varphi) = 0$ and the infimum is attained if and only if φ is periodic. Also $\tau(\varphi)$ is positive and the infimum is attained if and only if φ is pseudo-Anosov. Eventually, $\tau(\varphi)$ is not attained if and only if φ is reducible and of infinite order. Furthermore, if $\tau(\varphi) = 0$ and $\tau(\varphi)$ is not attained, then then there exists $\mu \in \mathbb{Z}_{\geq 1}$ bounded above by a constant determined by h such that ϕ^μ is a multi-twist.

2.2 Monodromy homomorphisms. Suppose that $B = \Gamma \backslash \mathbb{H}^2$ is a hyperbolic surface of type (g, n) with some lattice $\Gamma \leq \mathrm{Aut}(\mathbb{H}^2)$. Consider a holomorphic map $F : B \to \mathcal{M}_h$ and the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$. We introduce the induced *monodromy homomorphisms.*

Firstly, the map $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ induces a group homomorphism $F_\Gamma : \Gamma \to \mathrm{Mod}_h$ such that $\widetilde{F} \circ \phi = F_\Gamma(\phi) \circ \widetilde{F}$, for every $\phi \in \Gamma$. When $F(t)$ has non-identical automorphisms (i.e. $F(t)$ is symmetric) for some $t \in B$, the homomorphism F_Γ is not necessarily unique. There are at most $(2g + n)^{84(h-1)}$ many possibilities of such a homomorphism F_Γ.

Secondly, fixing a base point $t \in B$ and lifting it to some $\widetilde{t} \in \mathbb{H}^2$, we obtain a standard group isomorphism $\rho_{t,\widetilde{t}} : \pi_1(B, t) \to \Gamma$ as follows. A loop $\gamma \subset B$ based at t is lifted to the path in $\mathbb{H}^2$ joining $\widetilde{t}$ and $\rho_{t,\widetilde{t}}([\gamma]) \cdot \widetilde{t}$.

Eventually, $F_* := F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$ is called a *monodromy homomorphism* of F. A different choice of $\widetilde{t}$ may change the monodromy homomorphism by a conjugacy.

The monodromy homomorphism enables us to reformulate and slightly improve the most important property of the holomorphic map F, say being distance-decreasing for the intrinsic Kobayashi distances, as follows.

Proposition 2.2. *Let $\gamma \subset B$ be a loop based at t. Then $(1/2)l_B(\gamma) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), F_*([\gamma]) \cdot \widetilde{F}(\widetilde{t})\Big)$.*

Proof. By definition, we get

$$\begin{aligned}\frac{1}{2}l_B(\gamma) &= \frac{1}{2}d_{\mathbb{H}^2}\Big(\widetilde{t}, \rho_{t,\widetilde{t}}([\gamma])(\widetilde{t})\Big) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), \widetilde{F} \circ \rho_{t,\widetilde{t}}([\gamma_i])(\widetilde{t})\Big) \\ &= d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), \big(F_\Gamma \circ \rho_{t,\widetilde{t}}([\gamma])\big) \circ \widetilde{F}(\widetilde{t})\Big) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), F_*([\gamma]) \cdot \widetilde{F}(\widetilde{t})\Big).\end{aligned}$$

□

Proposition 2.3. *Let $\gamma \subset B$ be a loop based at t and $\gamma' \subset B$ be a free loop homotopic to γ. Then $(1/2)l_B(\gamma') \geq \tau(F_*([\gamma]))$.*

Proof. Let $H : [0,1] \times [0,1] \to B$ be the homotopy between γ and γ' such that $H(0,\cdot) = \gamma(\cdot)$, $H(0,0) = H(0,1) = t$ and $H(1,\cdot) = \gamma'(\cdot)$. Based on the path $H(\cdot,0)$ joining t to $t' := H(1,0)$, we obtain a lift of t', denoted by $\widetilde{t'}$. Consider the new monodromy homomorphism $F'_* = F_\Gamma \circ \rho_{t',\widetilde{t'}} : \pi_1(B,t') \to \mathrm{Mod}_h$. Since $F'_*([\gamma']) = F_*([\gamma]) =: \phi$, by Proposition 2.2, we have

$$\frac{1}{2}l_B(\gamma') \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), F'_*([\gamma']) \cdot \widetilde{F}(\widetilde{t'})\Big) \geq \tau(\phi).$$

□

Corollary 2.4. *Each peripheral monodromy ϕ satisfies $\tau(\phi) = 0$.*

Proof. The corollary follows from Proposition 2.3. □

2.3 Essentially purely pseudo-Anosov monodromy. We introduce the following hypothesis on the monodromy homomorphism (see also [Rei06]).

Definition 2.5. Let B be an oriented hyperbolic surface of type (g,n) and $F : B \to \mathcal{M}_h$ be a holomorphic map. We say that a monodromy homomorphism $F_* : \pi_1(B,t) \to \mathrm{Mod}_h$ of F is *essentially purely pseudo-Anosov* if for each non-trivial non-peripheral class $[\gamma] \in \pi_1(B,t)$ the image $F_*([\gamma])$ is pseudo-Anosov.

Isometrically immersed holomorphic curves are specific examples whose monodromy homomorphisms are essentially purely pseudo-Anosov. In fact, as mentioned in [EM11], every closed geodesic in $\mathcal{M}_h$ is the unique loop of minimal length in its homotopy class. We include a proof for the sake of completeness.

Theorem 2.6. *Each monodromy homomorphism of a Teichmüller curve is essentially purely pseudo-Anosov.*

Proof. Let $F : B \to \mathcal{M}_h$ be a holomorphic isometric immersion and $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ be a lift of F which is an isometric embedding. Fixing $t \in B$ and lifting it to $\widetilde{t} \in \mathbb{H}^2$, we take a monodromy homomorphism $F_* = F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B,t), \mathrm{Mod}_h)$.

Let $\gamma \subset B$ be a non-trivial non-peripheral loop based at t. Suppose that $\gamma' \subset B$ is a closed geodesic homotopic to γ. Consider the homotopy $H : [0,1] \times [0,1] \to B$ such that $H(0,\cdot) = \gamma(\cdot)$, $H(0,0) = H(0,1) = t$ and $H(1,\cdot) = \gamma'(\cdot)$. Based on the path $H(\cdot,0)$ joining t to some $t' \in \Gamma'$, we obtain a monodromy homomorphism $F'_* = F_\Gamma \circ \rho_{t',\widetilde{t'}} \in \mathrm{Hom}(\pi_1(B,t'), \mathrm{Mod}_h)$ such that $F_*([\gamma]) = F'_*([\gamma']) =: \phi$. By Proposition 2.2, we get

$$\frac{N}{2}l_B(\gamma') = \frac{1}{2}d_{\mathbb{H}^2}\Big(\widetilde{t'}, \rho_{t',\widetilde{t'}}([\gamma'])^N(\widetilde{t'})\Big) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big),$$

for any integer $N \geq 1$.

We claim that $\tau(\phi) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) = \frac{1}{2}l_B(\gamma')$. Indeed, assume that there exists $\widetilde{q} \in \mathcal{T}_h$ such that $d_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big) < d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big)$. Suppose that N is large enough such that $N\Big(d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) - d_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big)\Big) > 2d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big)$. Therefore, we get

$$\begin{aligned}\frac{N}{2}l_B(\gamma') &= d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big) \leq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big) + d_{\mathcal{T}}\Big(\widetilde{q}, \phi^N \cdot \widetilde{q}\Big) + d_{\mathcal{T}}\Big(\phi^N \cdot \widetilde{q}, \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big) \\ &\leq 2d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big) + Nd_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big) < Nd_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) = \frac{N}{2}l_B(\gamma'),\end{aligned}$$

a contradiction. Hence, the translation distance of the monodromy along any non-trivial non-peripheral loop is positive. □

The following proposition shows that being essentially purely pseudo-Anosov is a sufficiently strong hypothesis on the monodromy homomorphism of a holomorphic map.

Proposition 2.7. *Let $B = \Gamma\backslash\mathbb{H}^2$ be an oriented hyperbolic surface of type (g, n). Let $F : B \to \mathcal{M}_h$ be a holomorphic map with an essentially purely pseudo-Anosov monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), Mod_h)$. Then*

(i) F is non-constant;

(ii) the monodromy homomorphism is injective;

(iii) each peripheral monodromy is of infinite order;

(iv) peripheral monodromies of any two cusps are not disjoint along any geodesic segment κ between the boundaries of their cusp regions;

(v) $\mathrm{sys}(B) \geq 2\log 2/(12h - 12)$.

Proof. For (i), we notice that there exists at least one non-trivial non-peripheral element in $\pi_1(B, t)$. For (ii), it suffices to show that each peripheral element cannot be represented by the identity. Take the group presentation

$$\pi_1(B, t) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid \prod_i [a_i, b_i] \prod_j c_j = 1\rangle.$$

with loops $a_1, \ldots, a_g$, $b_1, \ldots, b_g$,$c_1, \ldots, c_n$ at t. Consider a positive power of a peripheral generating loop, say c_j^r with $j = 1, \ldots, n$. When $g \geq 1$ and $n \geq 1$, then $[c_j^r, a_1] \neq 1$ is non-peripheral. When $n \geq 2$, take $j' \neq j$ and then $[c_j^r, c_{j'}] \neq 1$ is non-peripheral. In both cases, we get $F_*(c_j^r) \neq 1$. For (iii), a peripheral monodromy ϕ must be of infinite order due to the injectivity of the monodromy homomorphism.

For (iv), let U_1, U_2 be two distinct cusp regions of B linked by a geodesic segment κ. Set $t_0 = \partial U_1 \cap \kappa$ and take an arbitrary path γ joining t to t_0. The loop along $\gamma \cup \partial U_1$ based at t that goes once around U_1 clockwise is denoted by γ_1 and its monodromy is denoted by ϕ_1. The loop along $\gamma \cup \kappa \cup \partial U_2$ based at t that goes once around U_2 clockwise is denoted by γ_2 and its monodromy is denoted by ϕ_2. By Corollary 2.4, some power $\phi_1^{\mu_1}$ is the multi-twist along a multi-curve $\boldsymbol{\alpha_1}$ and some power $\phi_2^{\mu_2}$ is the multi-twist along a multi-curve $\boldsymbol{\alpha_2}$. Assume that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is a set of disjoint simple closed curves. We notice that $\gamma_1^{2\mu_1} \cdot \gamma_2^{2\mu_2}$ is non-peripheral of which the monodromy is reducible, which is a contradiction. The parameters 2 in $\gamma_1^{2\mu_1} \cdot \gamma_2^{2\mu_2}$ cannot be replaced with 1. Indeed, when $(g, n) = (0, 3)$ and $\mu_1 = \mu_2 = 1$, $\gamma_1 \cdot \gamma_2$ is peripheral but $\gamma_1^2 \cdot \gamma_2^2$ not.

For (v), consider an essential closed geodesic $\gamma \subset B$. Then there exists a loop γ' based at t homotopic to γ, which is non-trivial and non-peripheral. By Proposition 2.3, we have $(1/2)l_B(\gamma) \geq \tau(F_*([\gamma']))$. Penner proved in [Pen91, p.444] the inequality $\tau(\phi) \geq \log 2/(12h - 12)$, for any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}_h$ (see also [FM11, Theorem 14.10]). Thus, $\mathrm{sys}(B) \geq 2\log 2/(12h - 12)$. □

We can apply Theorem A - (ii) and Theorem B to a holomorphic map $F : B \to \mathcal{M}_h$ with an essentially purely pseudo-Anosov monodromy homomorphism. Then F is a quasi-isometric immersion with parameters depending only on (g, n) and h. In addition, the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ restricted to any fundamental convex polygon D with exactly n ideal points is a quasi-isometric embedding.

Moreover, by Theorem C, there are only finitely many essentially purely pseudo-Anosov monodromy homomorphisms induced by Teichmüller curves of type (g, n) in $\mathcal{M}_h$, up to equivalence. When $n = 0$, this finiteness is a consequence of Bowditch's result [Bow09] which shows that there are only finitely many conjugacy classes of purely pseudo-Anosov surface subgroups of Mod_h of genus g (see also [Bow17; DF09]). In conclusion,

Conjecture 2.8. *There are only finitely many conjugacy classes of essentially purely pseudo-Anosov subgroups of* Mod_h *isomorphic to the fundamental group of* $\Sigma_{g,n}$.

We end with the proof of Corollary 1.7 as a consequence of the Rigidity Theorem in [IS88].

Proof of Corollary 1.7. This comes from Theorem 2.6, Proposition 2.7 and Theorem C. □

2.4 Mumford's compactness. Let $\epsilon > 0$ be an arbitrary real number. For the Teichmüller metric, the moduli space $\mathcal{M}_{g,n}$ has an infinite diameter. In [Mum71], however, Mumford introduced that the ϵ-thick part of the moduli space that consists of hyperbolic surfaces X with $\mathrm{sys}(X) \geq \epsilon$ is compact.

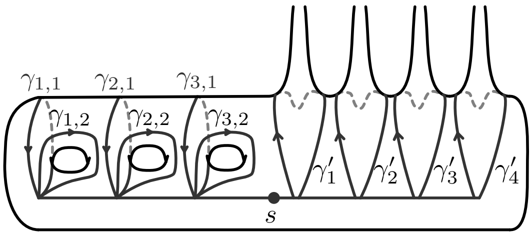
Let $\mathcal{T}_{g,n}^{\geq\epsilon}$ be the set of equivalent classes of marked hyperbolic surfaces whose systole is bounded below by ϵ. The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{T}_{g,n}$ preserves the systole and therefore we take the quotient space of the thick part, denoted by $\mathcal{M}_{g,n}^{\geq\epsilon}$. We have Mumford's compactness.

Theorem 2.9 (Mumford)**.** *The ϵ-thick part $\mathcal{M}_{g,n}^{\geq\epsilon}$ of the moduli space $\mathcal{M}_{g,n}$ is a compact subset.*

Form now on, we fix a base point $[(X_0, f_{X_0})]$ in $\mathcal{T}_{g,n}$ that is represented by a marked complete hyperbolic surface (X_0, f_{X_0}). We fix a base point $s \in \Sigma_{g,n}$ and fix the oriented loops

$$\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{g,1}, \gamma_{g,2}, \gamma'_1, \ldots, \gamma'_n \subset \Sigma_{g.n}$$

at s as in Figure 3.



$\Sigma_{3,4}$

$$\pi_1(\Sigma_{3,4}, s) = \prod_{i=1}^{3} [[\gamma_{i,1}], [\gamma_{i,2}]] \prod_{j=1}^{4} [\gamma'_j] = 1$$

Figure 3: The standard loops of $\Sigma_{3,4}$ generate the fundamental group $\pi_1(\Sigma_{g,n}, s)$.

Loops $\gamma_{i,j}$ and γ'_k, for $i = 1, \ldots, g$, $j = 1, 2$ and $k = 1, \ldots, n$, are called the *standard loops* of $\Sigma_{g,n}$. They meet the following conditions.

i. $\iota(\gamma_{i,1}, \gamma_{i,2}) = 1$, for each i, but the intersection number of any other two distinct loops is 0;

ii. γ'_j goes round the j-th puncture exactly once clockwise, for each j;

iii. the fundamental group $\pi_1(\Sigma_{g,n}, s)$ is generated by classes of these loops with the relation

$$\prod_{i=1}^{g} [[\gamma_{i,1}], [\gamma_{i,2}]] \prod_{j=1}^{n} [\gamma'_j] = 1.$$

In contrast, a collection of loops $\widehat{\gamma_{i,1}}$, $\widehat{\gamma_{i,2}}$ $\widehat{\gamma'_j} \subset \Sigma_{g,n}$, for $i = 1, \ldots, g$, $j = 1, \ldots, n$, based at some point $\hat{s} \in \Sigma_{g,n}$ satisfying conditions i, ii, iii implies an orientation preserving diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ such that $\phi(s) = \hat{s}$ and each $\phi(\gamma_{i,1})$ (resp. $\phi(\gamma_{i,2})$, $\phi(\gamma_j)$) is homotopic to $\widehat{\gamma_{i,1}}$ (resp. $\widehat{\gamma_{i,2}}$, $\widehat{\gamma'_j}$) relative to $\hat{s}$.

Set $t_0 = f_{X_0}(s) \in X_0$. The set $\{[f_{X_0}(\gamma)] \mid \gamma \text{ is a standard loop}\}$ forms a generating set of $\pi_1(X_0, t_0)$ such that the length of each $f_{X_0}(\gamma)$ is determined by (g, n). In other words, there exists a constant $N(g, n)$ that depends only on (g, n) such that

$$l_{X_0}(f_{X_0}(\gamma)) \leq N(g, n)$$

for each standard loop γ of $\Sigma_{g,n}$. We further notice the following lemma.

Lemma 2.10. *Suppose that (Y, f_Y) is a marked hyperbolic surface and $\psi : X_0 \to Y$ is a K-quasiconformal diffeomorphism. Then there exists an orientation preserving diffeomorphism $f'_Y : \Sigma_{g,n} \to Y$ such that, for each standard loop γ, the image $f'_Y(\gamma) \subset Y$ is homotopic to a loop of length bounded above by $N'(g, n, K)$ relative to $f'_Y(s)$, where $N'(g, n, K)$ depends only on g, n and K.*

Proof. We present $2g + n + [n = 0]$ free loops on $\Sigma_{g,n}$ where $[n = 0] = 1$ if $n = 0$ and $[n = 0] = 0$ if $n \neq 0$, denoted by $\delta_0, \ldots, \delta_{2g+n-1+[n=0]}$, that form a set of closed curves filling up the surface $\Sigma_{g,n}$.

- When $n = 0$, set

$$\begin{aligned} &\delta_0 = \gamma_{g,2}^{-1}, \ \delta_1 = \gamma_{1,1} \cdot \gamma_{g,2}^{-1} \cdot \gamma_{g,1}^{-1} \cdot \gamma_{g,2}, \\ &\delta_{2i} = \gamma_{i,2}^{-1}, \ \delta_{2i+1} = \gamma_{i+1,1} \cdot \gamma_{i,2}^{-1} \cdot \gamma_{i,1}^{-1} \cdot \gamma_{i,2}, && \text{for } i = 1, \ldots, g-1, \\ &\delta_{2g} = [\gamma_{1,1}, \gamma_{1,2}]. \end{aligned}$$

- When $g = 0$, set

$$\begin{aligned} \delta_0 &= \gamma'^{-1}_1 \cdot \gamma'^{-1}_g, \\ \delta_j &= \gamma'^{-1}_{j+1} \cdot \gamma'^{-1}_j, \qquad && \text{for } j = 1, \ldots, n-1. \end{aligned}$$

- When $g \geq 1$ and $n \geq 1$, set

$$\begin{aligned} &\delta_0 = \gamma_{1,1} \cdot \gamma'^{-1}_n, \\ &\delta_{2i-1} = \gamma^{-1}_{i,2},\ \delta_{2i} = \gamma_{i+1,1} \cdot \gamma^{-1}_{i,2} \cdot \gamma^{-1}_{i,1} \cdot \gamma_{i,2}, \qquad && \text{for } i = 1, \ldots, g-1, \\ &\delta_{2g-1} = \gamma^{-1}_{g,2},\ \delta_{2g} = \gamma'^{-1}_1 \cdot \gamma^{-1}_{g,2} \cdot \gamma^{-1}_{g,1} \cdot \gamma_{g,2}, \\ &\delta_{2g+j} = \gamma'^{-1}_{j+1} \cdot \gamma'^{-1}_j, && \text{for } j = 1, \ldots, n-1. \end{aligned}$$

Let Δ_Y be the union of geodesics homotopic to every $\psi \circ f_{X_0}(\delta_i)$. By Wolpert's Lemma,

$$l_Y(\Delta_Y) = \sum_{i=0}^{2g+n-1+[n=0]} L_{\delta_i}([(Y, f_Y)]) \leq e^{2K} \sum_{i=0}^{2g+n-1+[n=0]} L_{\delta_i}([(X_0, f_{X_0})]) \leq 4(2g+n)e^{2K} \cdot N(g,n).$$

Let $t_Y \in \Delta_Y$ be an arbitrary point. Therefore, there exists loops $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}}$ and $\widehat{\gamma'}_j$ at t_Y on Y, for $i = 1, \ldots, g$ and $j = 1, \ldots, n$, that satisfy conditions i, ii and iii, whose lengths are bounded above by

$$4 \cdot 4(2g+n)e^{2K} N(g,n) =: N'.$$

We conclude that there exists a diffeomorphism f'_Y sending $s \in \Sigma_{g,n}$ to t_Y such that, for each standard loop γ, the homotopy class of $f'_Y(\gamma)$ relative to t_Y is represented by a loop of length bounded above by N'. □

We aim at addressing the following question: given a hyperbolic surface X of type (g,n), does there exist an orientation preserving diffeomorphism $f'_X : \Sigma_{g,n} \to X$ whose image of each standard loop is homotopic to a loop of short length relative to $f'_X(s)$?

Consider a non-trivial mapping class $[\phi]$ represented by a diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$. Then, the marked hyperbolic surfaces (X, f_X) and $(X, f_X \circ \phi^{-1})$ are the same hyperbolic surface with different markings. Therefore, they possess desired diffeomorphisms for this question simultaneously. Using Mumford's compactness of $\mathcal{M}^{\geq\epsilon}_{g,n}$, we get the following theorem.

Theorem 2.11. *There exists a constant $N'' = N''(g,n,\epsilon)$ that depends only on (g,n), ϵ and satisfies the following statement. Given an arbitrary hyperbolic surface Y of type (g,n) with $\mathrm{sys}(Y) \geq \epsilon$, there exists an orientation preserving diffeomorphism $f'_Y : \Sigma_{g,n} \to Y$ such that the image $f'_Y(\gamma)$ of each standard loop $\gamma \subset \Sigma_{g,n}$ is homotopic to a loop of length bounded above by N'' relative to $f'_Y(s)$.*

Proof. By Mumford's compactness, the moduli space $\mathcal{M}^{\geq\epsilon}_{g,n}$ is compact and therefore there exists a lift $U \subset \mathcal{T}_{g,n}$ of $\mathcal{M}^{\geq\epsilon}_{g,n}$ containing $[X_0, f_{X_0}]$ whose diameter for the Teichmüller metric is bounded by a constant $D = D(g,n,\epsilon) > 0$ that depends only on (g,n) and ϵ. Suppose that f_Y is an arbitrary marking of Y so that $[(Y, f_Y)] \in \mathcal{T}^{\geq\epsilon}_{g,n}$. Then, there exists an orientation preserving diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ with $[(Y, f_Y \circ \phi^{-1})] \in U$. Moreover, there exists a K-quasiconformal diffeomorphism $\psi : X_0 \to Y$ homotopic to $(f_Y \circ \phi^{-1}) \circ f^{-1}_{X_0}$, where $e^{2K} \leq D$. Let $f'_Y : \Sigma_{g,n} \to Y$ be the diffeomorphism as in the previous lemma and take $N'' = N'(g,n,K)$. Therefore, for each standard loop γ on $\Sigma_{g,n}$, the image $f'_Y(\gamma)$ is homotopic to a loop of length bounded above by N'' relative to $f'_Y(s)$. □

3 Uniform boundedness for Parshin-Arakelov finiteness

Parshin-Arakelov finiteness investigates non-isotrivial holomorphic families of Riemann surfaces over a Riemann surface of finite type. Suppose that B is an oriented hyperbolic surface of type (g,n). A holomorphic family C/B that comes from a holomorphic map $F : B \to \mathcal{M}_h$ is non-isotrivial if and only of F is non-constant. Parshin-Arakelov finiteness claims that there are only finitely many non-isotrivial non-isomorphic families of closed Riemann surfaces of genus h over B. This also means that there are only finitely many non-constant holomorphic maps $F : B \to \mathcal{M}_h$.

Fix $\epsilon > 0$. In this section, however, we investigate a non-constant holomorphic map $F : B \to \mathcal{M}_h$ where B is an arbitrary, not fixed, oriented hyperbolic surface of type (g,n) with $\mathrm{sys}(B) \geq \epsilon$. In particular, we will prove Theorem C from the introduction.

Proposition 3.1 (Theorem 1 in [Shi97]). *Let B be an oriented hyperbolic surface of type (g,n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Suppose that $F_* : \pi_1(B,t) \to \mathrm{Mod}_h$ is a monodromy homomorphism induced by F with $t \in B$. Then there does not exist a set of non-homotopic disjoint simple closed curves $\{\alpha_1, \ldots, \alpha_{h'}\}$ on Σ_h such that the set of homotopy classes $\{[\alpha_1], \ldots, [\alpha_{h'}]\}$ is preserved by $F_*(g)$ for each $g \in \pi_1(B,t)$.*

This proposition is known as the *irreducibility* of holomorphic curves in the moduli space. It first appeared in [IS88] as part of the proof of the Parshin-Arakelov finiteness. Shiga later formally formulated this result in [Shi97]. For the case of $n = 0$, the irreducibility is a consequence of [DW07, Theorem 5.7] and is also proved using the maximum principle for subharmonic functions [McM00, Theorem 3.1]).

Theorem 3.2. *There exist a constant $N'' = N''(g,n,\epsilon)$ depending only on (g,n), ϵ and a compact subset $\mathcal{K}' = \mathcal{K}'(g,n,h,\epsilon) \subset \mathcal{M}_h$ depending only on $(g,n), h, \epsilon$ that satisfy the following statement. Let B be an oriented hyperbolic surface of type (g,n) such that $\mathrm{sys}(B) \geq \epsilon$ and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Then, there exists an orientation preserving diffeomorphism $f'_B : \Sigma_{g,n} \to B$ such that*

(i) f'_B sends the base point $s \in \Sigma_{g,n}$ to $t'_B \in B$ such that $F(t'_B) \in \mathcal{K}'$;

(ii) the image $f'_B(\gamma) \subset B$ of each standard loop $\gamma \subset \Sigma_{g,n}$ is homotopic to a loop relative to t'_B of length bounded above by N''.

Theorem 3.2 is a continuation of Theorem 2.11, in which the constant $N''(g,n,\epsilon)$ and the orientation preserving diffeomorphism f'_B are inherited. We start with a well-known lemma on closed hyperbolic surface, based on which we then prove that $\mathrm{sys}(F(t'_B))$ is bounded. The proof of Theorem 3.2 proceeds similarly to the proof of [McM00, Theorem 3.1].

Lemma 3.3 (Corollary 13.7 in [FM11]). *There exists a constant $\tau_h > 0$ depending only on h such that on any closed oriented hyperbolic surface X of genus h, if $\{\alpha_1, \ldots, \alpha_{h'}\}$ is the set of closed geodesics of length smaller than τ_h, then $\iota(\alpha_i, \alpha_j) = 0$ and $h' \leq 3h-3$.*

Proof of Theorem 3.2. We aim to show that $\mathrm{sys}(F(t'_B)) \geq \tau_h / N''^{3h-3}$ and therefore take $\mathcal{K}' = \mathcal{M}_h^{\geq \tau_h / N''^{3h-3}}$, which is a compact subset by Mumford's compactness. Suppose that $F(t'_B)$ has a hyperbolic representative (X, f_X) and assume that $\mathrm{sys}(X) < \tau_h / N''^{3h-3}$. By Lemma 3.3, let $\alpha_1, \ldots, \alpha_{h'}$ be closed geodesics on X of length smaller than τ_h, with $h' \leq 3h-3$.

By Wolpert's Lemma, images under f_X^{-1} of the shortest several of $\alpha_1, \ldots, \alpha_{h'}$ form a set of homotopy classes on Σ_h that is preserved by $F_*(g) \in \mathrm{Mod}_h$ for each $g \in \pi_1(B, t'_B)$. This contradicts with Proposition 3.1. □

Now, we provide the uniform boundedness for Parshin-Arakelov finiteness.

Theorem 3.4. *The subset*

$$\left\{ \boldsymbol{MO}(F) \middle| \begin{array}{l} B \text{ is an oriented hyperbolic surface of type } (g,n) \text{ such that } \mathrm{sys}(B) \geq \epsilon \\ F : B \to \mathcal{M}_h \text{ is a non-constant holomorphic map} \end{array} \right\} \in M_{g,n,h}$$

is finite, where the finiteness depends only on g, n, h and ϵ.

Proof. Let $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ be the lift of F. Consider a monodroy homomorphism $F_* = F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B,t), \mathrm{Mod}_h)$. By Theorem 2.11 and Theorem 3.2, there exists an orientation preserving diffeomorphism $f'_B : \Sigma_{g,n} \to B$ such that $f'_B(s) = t'_B$ and, for each standand loop $\gamma \subset \Sigma_{g,n}$, there exists a loop $\gamma_B \subset B$ homotopic to $f'_B(\gamma)$ relative to t'_B such that $l_B(\gamma_B) \leq N'' = N''(g,n,\epsilon)$. Besides, we have a compact subset $\mathcal{K}' = \mathcal{K}'(g,n,h,\epsilon) \subset \mathcal{M}_h$ such that $F(t'_B) \in \mathcal{K}'$.

When $t = t'_B$, it suffices to show that there are only finitely many possibilities of $F_*([\gamma_B]) \in \mathrm{Mod}_h$ for each standard loop γ. Since $\mathcal{K}'$ is compact, it has a bounded lift $\widetilde{\mathcal{K}'} \subset \mathcal{T}_h$ containing $\widetilde{F}(\widetilde{t})$. For each standard loop $\gamma \subset \Gamma_{g,n}$, by Proposition 2.2, the mapping class $g := F_*([\gamma_B]) \in \mathrm{Mod}_h$ has to be such that $d_{\mathcal{T}}(\widetilde{F}(\widetilde{t}), g \cdot \widetilde{F}(\widetilde{t})) \leq N''/2$. As the Teichmüller metric is proper, the subset $\mathrm{B}(\widetilde{\mathcal{K}'}, N''/2)$ is again compact. Since the mapping class group Mod_h acts properly discontinuously on $\mathcal{T}_h$, there are only finitely many choices of g.

In general, there exists an orientation preserving diffeomorphism $f : \Sigma_{g,n} \to B$ such that $f(s) = t$ and f is homotopic to f'_B. Let $H : [0,1] \times \Sigma_{g,n} \to B$ be the homotopy such that $H(0,\cdot) = f(\cdot)$ and $H(1,\cdot) = f'_B(\cdot)$. The path $H(\cdot, s)$ joining t to t'_B induces a new monodromy homomorphism $F'_* \in \mathrm{Hom}(\pi_1(B, t'_B), \mathrm{Mod}_h)$ such that $F_*([f(\gamma)]) = F'_*([f'_B(\gamma)])$ for each standard loop γ. Hence, there are only finitely many possibilities of $F_* \circ f_*$. □

4 Quasi-isometric rigidity

Suppose that B is a hyperbolic surface of type (g,n) and $F : B \to \mathcal{M}_h$ is a holomorphic map. In this section, we introduce the rigidity result, which claims that the holomorphic curve $F(B)$ is very similar to a Teichmüller curve.

4.1 From cusp region to end of moduli space. The moduli space $\mathcal{M}_h$ has only one end, meaning that for any compact set, there is exactly one unbounded component of the complement. In this subsection, we consider a hyperbolic cusp region U, i.e. the neighbourhood of a cusp bounded by a horocycle of length 2. Then we investigate a non-constant map $F : U \to \mathcal{M}_h$ that is distance-decreasing for 1/2 of the hyperbolic distance d_U on U and the Teichmüller distance $d_{\mathcal{M}}$ on $\mathcal{M}_h$.

The Dehn twist along a closed curve $\alpha \subset \Sigma_h$, denoted by τ_α, is a diffeomorphism of Σ_h and represents a mapping class $T_\alpha \in \mathrm{Mod}_h$. Let $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ be a multi-curve, then a multi-twist along $\boldsymbol{\alpha}$ is a product of the form $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ with each $r_i \in \mathbb{Z} \setminus \{0\}$. In particular, a power of positive or negative Dehn twist is a multi-twist in our discussion.

Lemma 4.1. *Let $\widetilde{\gamma_1}, \widetilde{\gamma_2} \subset \mathbb{H}^2$ be disjoint geodesics and $k \geq 3$ be an integer. Suppose that ϕ is a hyperbolic isometry along $\widetilde{\gamma_1}$ whose translation length is equal to l such that $\widetilde{\gamma_2} \cap \phi(\widetilde{\gamma_2}) = \emptyset$. Let $\widetilde{p_1}, \widetilde{p_2} \in \mathbb{H}^2$ be arbitrary points partitioned by both $\widetilde{\gamma_2}$ and $\phi^k(\widetilde{\gamma_2})$. Then $d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq l$.*

Proof. Take the (unique) geodesic segment β_i perpendicular to both $\widetilde{\gamma_1}$ and $\phi^i(\widetilde{\gamma_2})$, for $i = 1, 2$. The hyperbolic plane is separated by $\widetilde{\gamma_1}$, $\widetilde{\gamma_2}$, $\phi(\widetilde{\gamma_2})$, $\phi^2(\widetilde{\gamma_2})$, $\phi^k(\widetilde{\gamma_2})$ and β_1, β_2 into 8 pieces. Therefore, the geodesic segment joining $\widetilde{p_1}$ to $\widetilde{p_2}$ goes cross β_1 and β_2. Hence the distance between $\widetilde{p_1}$ and $\widetilde{p_2}$ is at least the distance between β_1 and β_2, which is equal to l. □

Lemma 4.2 (Lemma 4.2 in [Iva92], cf. Exposé 6, Section VII in [FLP12])**.** *Let $\tau = \tau_{\alpha_1}^{r_1} \circ \cdots \circ \tau_{\alpha_m}^{r_m}$ be a multi-twist diffeomorphism along $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_m\}$. Then for all closed curves $\gamma_1, \gamma_2 \subset \Sigma_h$, we have*

$$\iota(\tau(\gamma_1), \gamma_2) \geq \sum_{i=1}^{m} (|r_i| - 2)\, \iota(\gamma_1, \alpha_i)\, \iota(\gamma_2, \alpha_i) - \iota(\gamma_1, \gamma_2).$$

In particular, for any multi-twist diffeomorphism τ along $\boldsymbol{\alpha}$ and closed curve γ intersecting $\boldsymbol{\alpha}$ at least once, we have $\iota(\tau^3(\gamma), \gamma) \geq \sum_{i=1}^{m} (|3r_i| - 2)\, \iota(\gamma, \alpha_i)^2 \geq 1$. The following lemma is inspired by this observation.

Lemma 4.3. *Let $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ be a multi-twist along $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ and $\gamma \subset \Sigma_h$ be a simple closed curve such that $\iota(\boldsymbol{\alpha}, \gamma) \coloneqq \sum_i |r_m|\, \iota(\alpha, \gamma) \geq 1$. Then, given $[(X, f_X)] \in \mathcal{T}_h$, we have*

$$L_\gamma([(X, f_X)]) + L_\gamma(T^4 \cdot [(X, f_X)]) \geq \frac{1}{3} L_{\alpha_i}([(X, f_X)])$$

for all $i = 1, \dots, m$.

Proof. Without loss of generality, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$. Suppose that T is represented by a multi-twist diffeomorphism τ. We have

$$L_\gamma(T^4 \cdot [(X, f_X)]) = L_{\tau^{-4}(\gamma)}([(X, f_X)])$$

since $T \cdot [(X, f_X)] = [(X, f_X \circ \tau^{-1})]$.

Consider a universal covering $\iota : \mathbb{H}^2 \to X$ such that the horizontal geodesic $\widetilde{\gamma_1} \subset \mathbb{H}^2$ is a lift of the (unique) geodesic homotopic to $f_X(\gamma) \subset X$ (see Figure 4). We suppose that $\widetilde{\gamma_1}$ is oriented toward the left. Proceeding from $0 \in \mathbb{H}^2$, the first lift of some $f_X(\alpha_u)$ that intersects $\widetilde{\gamma_1}$ is denoted by $\widetilde{\alpha_1}$ and the second lift of some $f_X(\alpha_v)$ is denoted by $\widetilde{\alpha_2}$. Going along the opposite direction, the first lift of some $f_X(\alpha_x)$ is denoted by $\widetilde{\alpha_{-1}}$ and the second lift of some $f_X(\alpha_y)$ is denoted by $\widetilde{\alpha_{-2}}$. Suppose that $\widetilde{\alpha_1}$ and $\widetilde{\alpha_{-1}}$ are oriented upward.

Fix $i = 1, \dots, m$. Without loss of generality, we further assume that $u = i$.

The closed geodesic homotopic to $f_X(\alpha_u) \subset X$ is interpreted by a hyperbolic isometry $\phi_1 \in \mathrm{Isom}(\mathbb{H}^2)$ of which $\widetilde{\alpha_1}$ is the axis, whose translation length is $L_{\alpha_u}([(X, f_X)])$. Therefore, the twist $T_{\alpha_u}^{r_u}$ is interpreted by $\phi_1^{r_u}$ acting on the left. Similarly, the closed geodesic homotopic to $f_X(\alpha_x) \subset X$ is interpreted by a hyperbolic isometry $\phi_{-1} \in \mathrm{Isom}(\mathbb{H}^2)$ of which $\widetilde{\alpha_{-1}}$ is the axis, whose translation length is $L_{\alpha_x}([(X, f_X)])$. Therefore, the twist $T_{\alpha_x}^{r_x}$ is interpreted by $\phi_{-1}^{-r_x}$ acting on the right.

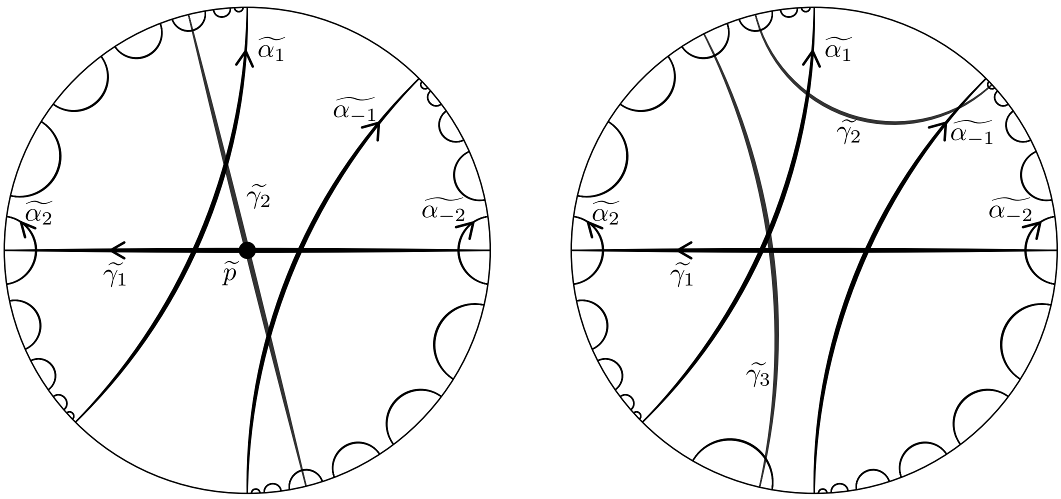


Figure 4: Lifts of γ and $T^4(\gamma)$ given a multi-twist T along $\boldsymbol{\alpha}$.

Suppose that the twists within T along α_u and α_x have the same direction. Without lose of generality, we assume that $r_u > 0$ and $r_x > 0$. There exists a lift $\widetilde{\gamma_2}$ of the (unique) geodesic homotopic to $f_X(\tau^{-4}(\gamma)) \subset X$ which connects two boundary points partitioned by both $\phi_1^{4r_u}(\widetilde{\alpha_2})$ and $\phi_{-1}^{-4r_x}(\widetilde{\alpha_{-2}})$. Let $\widetilde{p} \in \mathbb{H}^2$ be the intersection $\widetilde{\gamma_1} \cap \widetilde{\gamma_2}$. Let ψ_1, ψ_2 be hyperbolic isometries along $\widetilde{\gamma_1}$, $\widetilde{\gamma_2}$ corresponding to $f_X(\gamma)$, $f_X(\tau^{-4}(\gamma))$ respectively. Therefore,

$$\begin{aligned} 3\Big(L_\gamma([(X,f_X)]) + L_\gamma(T^4 \cdot [(X,f_X)])\Big) &= 3\Big(L_\gamma([(X,f_X)]) + L_{\tau^{-4}(\gamma)}([(X,f_X)])\Big) \\ &= d_{\mathbb{H}^2}(\widetilde{p}, \psi_1^3(\widetilde{p})) + d_{\mathbb{H}^2}(\widetilde{p}, \psi_2^3(\widetilde{p})). \end{aligned}$$

Meanwhile, $\psi_1^3(\widetilde{p})$ and $\psi_2^3(\widetilde{p})$ are partitioned by both $\widetilde{\alpha_2}$ and $\phi_1^{4r_u}(\widetilde{\alpha_2})$. The desired inequality follows from Lemma 4.1.

Suppose that twists within T along α_u and α_x have different directions. Without loss of generality, we assume that $r_u > 0$ and $r_x < 0$. In this case, we have $\iota(\boldsymbol{\alpha}, \gamma) \geq 2$. Take the lift $\widetilde{\gamma_2}$ of the geodesic homotopic to $f_X(\tau^{-4}(\gamma)) \subset X$ which connects two boundary points partitioned by both $\phi_1^{4r_u}(\widetilde{\alpha_2})$ and $\phi_{-1}^{-4r_x}(\widetilde{\alpha_{-2}})$. Note that $\widetilde{\gamma_2}$ does not intersect $\widetilde{\gamma_1}$. The geodesic $\widetilde{\gamma_3} = \phi_1^{-1}(\widetilde{\gamma_2})$ is another lift of the geodesic homotopic to $f_X(\tau^{-4}(\gamma))$, which connects two boundary points that are partitioned by $\phi_1^{4r_u-1}(\widetilde{\alpha_2})$ and $\phi_1^{-1}(\widetilde{\alpha_{-1}})$. Hence $\widetilde{\gamma_3}$ intersects $\widetilde{\gamma_1}$, where the intersection is denoted by $\widetilde{p}$. Let ψ_1, ψ_3 be hyperbolic isometries along $\widetilde{\gamma_1}$, $\widetilde{\gamma_3}$ corresponding to $f_X(\gamma)$, $f_X(\tau^{-4}(\gamma))$ respectively. Therefore, $\psi_1^3(\widetilde{p})$ and $\psi_3^3(\widetilde{p})$ are partitioned by both $\widetilde{\alpha_2}$ and $\phi_1^{4r_u-1}(\widetilde{\alpha_2})$. Again, the desired inequality follows from Lemma 4.1. □

The multi-twist of a hyperbolic surface that provides a very slight deformation of the hyperbolic structure should be along a set of very short closed geodesics. We formulate this property in Proposition 4.4 and Proposition 4.6, which should be well-known.

Proposition 4.4. *Given $\mu \in \mathbb{Z}_{>0}$, there exists a constant $l_{max} = l_{max}(h, \mu)$ that depends only on h and μ such that, for any multi-twist $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ and $[(X, f_X)] \in \mathcal{T}_h$, if*

$$d_{\mathcal{T}}([(X,f_X)], T \cdot [(X,f_X)]) \leq 2\mu$$

then

$$L_{\alpha_i}([(X,f_X)]) \leq l_{max}$$

for all $i = 1, \ldots, m$.

Proof. Without loss of generality, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$. There exists a geodesic pants decomposition $\mathcal{P}_X = \{\gamma_i\}$ of X with each $l_X(\gamma_i)$ bounded above by Bers' constant (cf. Theoem 12.8 in [FM11]). More precisely, $l_X(\gamma_i) \leq 21(h-1)$ for each $\gamma_i \in \mathcal{P}_X$. Suppose that $\alpha_{X,i} \subset X$ is the (unique) geodesic homotopic to $f_X(\alpha_i)$, for each $\alpha_i \in \boldsymbol{\alpha}$, and set $\boldsymbol{\alpha}_X = \{\alpha_{X,i} \mid i = 1, \ldots, m\}$. There are two cases to consider for all $i = 1, \ldots, m$.
Case 1: $\alpha_{X,i} \in \mathcal{P}_X$**.** Then, $L_{\alpha_i}([(X,f_X)]) = l_X(\alpha_{X,i}) \leq 21(h-1)$.
Case 2: $\alpha_{X,i} \notin \mathcal{P}_X$**.** Then, there exists a simple closed curve $\gamma \subset \Sigma_h$ such that $\iota(\boldsymbol{\alpha}, \gamma) \geq 1$ and $L_\gamma([(X,f_X)]) \leq 21(h-1)$. By Wolpert's Lemma,

$$L_\gamma(T^4 \cdot [(X,f_X)]) \leq \exp\{2 \cdot d_{\mathcal{T}}([(X,f_X)], T^4 \cdot [(X,f_X)])\} \cdot L_\gamma([(X,f_X)]) \leq e^{16\mu} \cdot L_\gamma([(X,f_X)]).$$

Hence, by Lemma 4.3, we get

$$\begin{aligned} L_{\alpha_i}([(X, f_X)]) &\le 3\Big(L_\gamma([(X, f_X)]) + L_\gamma(T^4 \cdot [(X, f_X)])\Big) \\ &\le 3L_\gamma([(X, f_X)])(1 + e^{16\mu}) \le 63(h-1)(1 + e^{16\mu}) =: l_{\max}(h, \mu). \end{aligned}$$

□

We shall consider Fenchel-Nielsen coordinates for $\mathcal{T}_h$ associated with a pants decomposition $\mathcal{P}$ of Σ_h. Let $\mathcal{P} = \{C_i\}$ be a set of closed curves on Σ_h. The *length parameter* of C_i is denoted by $l_i = L_{C_i}([(X, f_X)])$. However, the *twist parameter* θ_i is chosen to be proportional along C_i so that a positive Dehn twist along C_i changes the twist parameter by adding 2π. For $[(X, f_X)], [(X', f_{X'})] \in \mathcal{T}_h$, the *Fenchel-Neilsen distance* with respect to $\mathcal{P}$ is defined by

$$d_{\mathrm{FN},\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) = \sup_i \max\{|\log l_i - \log l'_i|, |l_i\theta_i - l'_i\theta'_i|\}$$

where $[(X, f_X)], [(X', f_{X'})]$ have Fenchel-Nielsen coordinates $c = (l_i, \theta_i)_i$ and $c' = (l'_i, \theta'_i)_i$ respectively. Fenchel-Nielsen distance is introduced and investigated in [ALPSS11]. In fact, we have the following quasi-isometric relation between Fenchel-Nielsen distance and Teichmüller distance.

Proposition 4.5. *Let $[(X, f_X)]$, $[(X', f_{X'})] \in \mathcal{T}_h$ be arbitrary and $\mathcal{P} = \{C_i\}$ be a pants decomposition of Σ_h such that $[(X, f_X)]$, $[(X', f_{X'})]$ have Fenchel-Nielsen coordinates $c = (l_i, \theta_i)_i$, $c' = (l'_i, \theta'_i)_i$ respectively. Suppose that, for some constants $N_1, N_2 > 0$, we have $l_i, l'_i \le N_1$ for all $i = 1, 2, \ldots$ and $d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \le N_2$. Then,*

$$d_{\mathcal{T}}([(X, f_X)], [(X', f_{X'})]) \le d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \cdot N_3(N_1, N_2)$$

and

$$d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \le d_{\mathcal{T}}([(X, f_X)], [(X', f_{X'})]) \cdot N_4(N_1)$$

where the constant $N_3(N_1, N_2)$ depends on N_1 and N_2, the constant $N_4(N_1)$ depends only on N_1.

Proof. The first inequality comes from Proposition 8.4 in [ALPSS11] and the second inequality comes from Corollary 8.8 in [ALPSS11]). □

We improve Proposition 4.4 using Proposition 4.5.

Proposition 4.6. *Given $\mu \in \mathbb{Z}_{>0}$, there exists a constant $K_1 = K_1(h, \mu)$ that depends only on h and μ such that, for any multi-twist $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ and $[(X, f_X)] \in \mathcal{T}_h$, if*

$$d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)]) \le 2\mu$$

then

$$L_{\alpha_i}([(X, f_X)]) \le K_1 d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])$$

for all $i = 1, \ldots, m$.

Proof. As in Proposition 4.4, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$ and let $\boldsymbol{\alpha}_X$ be the set of geodesics on X homotopic to $f_X(\alpha_i)$ for every $\alpha_i \in \boldsymbol{\alpha}$. Therefore, there exists a(nother) geodesic pants decomposition $\mathcal{P}_X = \{\gamma_i\}$ of X such that

- $\boldsymbol{\alpha}_X \subseteq \mathcal{P}_X$;
- $l_X(\gamma) \le \mathrm{Bers}(\boldsymbol{\alpha}_X)$ for each $\gamma \in \mathcal{P}_X$, where $\mathrm{Bers}(\boldsymbol{\alpha}_X)$ is a variation of Bers' constant that depends only on h and lengths of every geodesics in $\boldsymbol{\alpha}_X$, therefore depends only on h and μ.

Set $C_i = f_X^{-1}(\gamma_i)$ and then $\mathcal{P} = \{C_i\}$ is a pants decomposition of Σ such that $L_{C_i}([(X, f_X)]) \le \mathrm{Bers}(\boldsymbol{\alpha}_X)$, for each C_i. By Wolpert's Lemma, we further have

$$L_{C_i}(T \cdot [(X, f_X)]) \le \exp\{2 \cdot d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])\} \cdot L_{C_i}([X, f_X]) \le e^{4\mu}\, \mathrm{Bers}(\boldsymbol{\alpha}_X).$$

Therefore, by Proposition 4.5, there exists a constant K_1 depending only on $\mathrm{Bers}(\boldsymbol{\alpha}_X)$ and $e^{4\mu}\, \mathrm{Bers}(\boldsymbol{\alpha}_X)$ such that

$$L_{\alpha_i}([(X, f_X)]) \le \frac{1}{2\pi} d_{\mathrm{FN},\mathcal{P}}([(X, f_X)], T \cdot [(X, f_X)]) \le K_1 d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])$$

for all $i = 1, \ldots, m$. □

From now on, we consider a hyperbolic cusp region U and a non-constant map $F : U \to \mathcal{M}_h$ that is distance-decreasing for $(1/2)d_U$ on U and $d_{\mathcal{M}}$ on $\mathcal{M}_h$.

Suppose that $U = \langle g \rangle \backslash \mathcal{B}$ with $\mathcal{B} \subset \mathbb{H}^2$ a horoball and $g \in \mathrm{Aut}(\mathbb{H}^2)$ a parabolic isometry. Therefore, the map F can be lifted to a map $\widetilde{F} : \mathcal{B} \to \mathcal{T}_h$. Let $\phi \in \mathrm{Mod}_h$ be such that, given a generating loop $\gamma \subset U$ based at $p \in U$ and a lift $\widetilde{p} \in \mathcal{B}$ of p, one can lift $F(\gamma)$ to a path joining $\widetilde{F}(\widetilde{p})$ to $\phi \cdot \widetilde{F}(\widetilde{p})$. This mapping class ϕ should satisfy the inequality $\epsilon/2 \geq d_{\mathcal{T}}(\widetilde{F}(\widetilde{p}), \phi \cdot \widetilde{F}(\widetilde{p}))$, where $0 < \epsilon \leq 2$ and the horocycle $H_\epsilon \subset U$ of length ϵ contains p. Such a mapping class is called the monodromy of F.

We suppose that a power ϕ^μ is exactly a multi-twist T along $\boldsymbol{\alpha}$. Each point $p \in U$ determines the unique horocycle $H_\varepsilon \subset U$ such that $p \in H_\varepsilon$. Each lift $\widetilde{p} \in \mathcal{B}$ of p determines the geodesic length $l = L_{\alpha_1}(\widetilde{F}(\widetilde{p}))$. We associate the length amount ϵ and the length amount l to show that F is a quasi-isometric embedding.

Theorem 4.7. *Given $\epsilon > 0$, there exists $K_2 = K_2(h, \mu, \epsilon)$ that depends only on h, μ, ϵ and satisfies the following statement. Suppose that $\mathrm{sys}(F(p_{max})) \geq \epsilon$ for some $p_{max} \in \partial U$. Then, we have*

$$\frac{1}{2}d_U(p_1, p_2) \geq d_{\mathcal{M}}(F(p_1), F(p_2)) \geq \frac{1}{2}d_U(p_1, p_2) - K_2$$

for any pair of points (p_1, p_2) in U.

Proof. Let $p_{\max} \in \partial U$ and $p \in H_\varepsilon \subset U$ be arbitrary with $\varepsilon \leq 2$. Take a lift $\widetilde{p_{\max}} \in \mathcal{B}$ of $p_{\max}$ and a lift $\widetilde{p} \in \mathcal{B}$ of p such that $d_{\mathcal{M}}(F(p_{\max}), F(p)) = d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_{\max}}), \widetilde{F}(\widetilde{p}))$. For convenience, we set $q_{\max} = F(p_{\max})$, $q = F(p)$, $\widetilde{q_{\max}} = \widetilde{F}(\widetilde{p_{\max}})$ and $\widetilde{q} = \widetilde{F}(\widetilde{p})$. By Wolpert's Lemma, Proposition 4.6 and the triangle inequality in (U, d_U), we have

$$\begin{aligned}
d_{\mathcal{M}}(q, q_{\max}) = d_{\mathcal{T}}(\widetilde{q}, \widetilde{q_{\max}}) &\geq \frac{1}{2}\log\frac{L_{\alpha_1}(\widetilde{q_{\max}})}{L_{\alpha_1}(\widetilde{q})} \\
&\geq \frac{1}{2}\log\frac{\mathrm{sys}(\widetilde{q_{\max}})}{K_1(h,\mu)d_{\mathcal{T}}(\widetilde{q}, T\cdot\widetilde{q})} \geq \frac{1}{2}\log\frac{\mathrm{sys}(q_{\max})}{K_1(h,\mu)\mu\epsilon/2} = \frac{1}{2}\Big\{\log\frac{\mathrm{sys}(q_{\max})}{K_1(h,\mu)\mu} + \log\frac{2}{\epsilon}\Big\} \\
&\geq \frac{1}{2}d_U(p, p_{\max}) - K_2'
\end{aligned}$$

where $K_2' = 1 - (1/2)\log\mathrm{sys}(q_{\max}) + (1/2)\log K_1(h,\mu)\mu$.

In general, let $p_1, p_2 \in U$ be arbitrary. Set $q_1 = F(p_1)$, $q_2 = F(p_2)$ and take the corresponding horocycles $H_{\varepsilon_1} \ni p_1$, $H_{\varepsilon_2} \ni p_2$. Using the above inequality and triangle inequalities in both (U, d_U) and $(\mathcal{M}_h, d_{\mathcal{M}})$, we conclude that

$$\begin{aligned}
\frac{1}{2}d_U(p_1, p_2) &\geq d_{\mathcal{M}}(q_1, q_2) \\
&\geq |d_{\mathcal{M}}(q_1, q_{\max}) - d_{\mathcal{M}}(q_2, q_{\max})| \\
&\geq \frac{1}{2}|d_U(p_1, p_{\max}) - d_U(p_2, p_{\max})| - K_2' \\
&\geq \frac{1}{2}\max\Big\{\Big(\log\frac{2}{\varepsilon_1} - 2\Big) - \Big(\log\frac{2}{\varepsilon_2} + 2\Big), \Big(\log\frac{2}{\varepsilon_2} - 2\Big) - \Big(\log\frac{2}{\varepsilon_1} + 2\Big)\Big\} - K_2' \\
&= \frac{1}{2}\max\Big\{\log\frac{\varepsilon_1}{\varepsilon_2}, \log\frac{\varepsilon_2}{\varepsilon_1}\Big\} - 2 - K_2' \\
&\geq \frac{1}{2}(d_U(p_1, p_2) - 2) - 2 - K_2' = \frac{1}{2}d_U(p_1, p_2) - 3 - K_2'.
\end{aligned}$$

□

4.2 Proof of Theorem A. Consider an oriented hyperbolic surface B of type (g, n), which has n cusps. Let U_i be the cusp region of the i-th cusp, which is of area 2 and bounded by a horocycle of length 2, for $i = 1, \ldots, n$. The complement is a compact hyperbolic surface with boundary, denoted by $B_{cp} \subset B$.

The proof of Theorem A - (i) is made up of two lemmata. The first lemma claims that the holomorphic map restricted to a cusp region U_i is a quasi-isometric embedding whose parameters depend not only on (g, n), h and $\mathrm{sys}(B)$ but also on $\mathrm{sys}(F(b))$ for an arbitrary point $b \in B_{cp}$. The second lemma claims that $\mathrm{sys}(F(b))$ is bounded uniformly for $b \in B_{cp}$. Theorem A - (ii) is a consequence of Theorem A - (i) due to the fact that $\mathrm{diam}(\overline{B_{cp}})$ has an upper bound based on $\mathrm{sys}(B)$.

Lemma 4.8. *Given $\epsilon > 0$, there exists a constant $K_3 = K_3(g, n, h, \epsilon)$ that depends only on (g, n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$ such that a peripheral monodromy of the i-th cusp is of infinite order, for some $i = 1, \dots, n$. Suppose that $\mathrm{sys}(B) \geq \epsilon$ and $\mathrm{sys}(F(b)) \geq \epsilon$ for some $b \in B_{cp}$. Then, we have*

$$\frac{1}{2} d_B(p_1, p_2) \geq d_{\mathcal{M}}(F(p_1), F(p_2)) \geq \frac{1}{2} d_B(p_1, p_2) - K_3$$

for each pair of points $(p_1, p_2) \in U_i \times U_i$.

Proof. Since F is holomorphic, it is automatically distance-decreasing for $(1/2)d_B$ on B and $d_{\mathcal{M}}$ on $\mathcal{M}_h$. Now regard the hyperbolic surface B as the union of the compact region B_{cp} and n more disjoint cusp regions $U_1, \dots, U_n$ each bounded by a horocycle of length 2. Select an arbitrary boundary point $p_{\max,i} \in \partial U_i$ and let $\mathrm{diam}\,(B_{cp})$ be the diameter of B_{cp}.

A peripheral monodromy of the i-th cusp, denoted by ϕ, is reducible, of infinite order and has no pseudo-Anosov reduced component. Therefore, some power ϕ^μ is identical on each component, where μ is bounded above by a constant determined by h. Hence, ϕ^μ is a multi-twist.

Suppose that $\mathrm{sys}(F(b)) \geq \epsilon$ for some $b \in B_{cp}$. Let (p_1, p_2) be a pair of points in U_i. By Theorem 4.7, the difference between $d_{\mathcal{M}}(F(p_1), F(p_2))$ and $(1/2)d_B(p_1, p_2)$ is bounded by a constant depending only on h, μ and a lower bound of $\mathrm{sys}(F(p_{\max,i}))$. By Mumford's compactness, the diameter $\mathrm{diam}\,(B_{cp})$ is bounded by a constant determined by (g, n) and ϵ. Wolpert's Lemma then shows a mutual dependence between $\mathrm{sys}(F(b))$ and $\mathrm{sys}(F(p_{\max,i}))$. Hence, the unique parameter used in the desired inequality depends only on (g, n), h and ϵ. □

Lemma 4.9. *Given $\epsilon > 0$, there exists a constant $K_4 = K_4(g, n, h, \epsilon)$ that depends only on (g, n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) such that $\mathrm{sys}(B) \geq \epsilon$ and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Then, for any $p \in B_{cp}$, we have*

$$\mathrm{sys}(F(p)) \geq K_4.$$

Proof. One can derive this from Theorem 3.2 and Wolpert's Lemma. □

4.3 Proof of Theorem B. It remains to investigate a pair of points in distinct cusp regions. Theorem B comes from the following lemma.

Lemma 4.10. *Given $\epsilon > 0$, there exists a constant $K_5 = K_5(g, n, h, \epsilon)$ that depends only on (g, n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) such that $\mathrm{sys}(B) \geq \epsilon$ and $D \subset \mathbb{H}^2$ be a fundamental convex polygon of B with exactly n ideal points. Let $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$. For some $i \neq j$, $i = 1, \dots, n$ and $j = 1, \dots, n$, if peripheral monodromies of the i-th and the j-th cusps are of infinite order and they are not disjointed along some geodesic segment $\kappa_{i,j} \subset B$ having a lift $\widetilde{\kappa_{i,j}} \subset D$, then we have*

$$\frac{1}{2} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_1}), \widetilde{F}(\widetilde{p_2})) \geq \frac{1}{4} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) - K_5 - l_B(\kappa_{i,j})$$

for each pair of points $(p_1, p_2) \in U_i \times U_j$, where $\widetilde{p_1} \in D$ is a lift of p_1 and $\widetilde{p_2} \in D$ is a lift of p_2.

Proof. Without loss of generality, we assume that $i = 1$ and $j = 2$. Suppose that $p_1 \in H_{\epsilon_1} \subset U_1$ where H_{ϵ_1} is the horocycle of length $0 < \epsilon_1 \leq 2$ within the cusp region U_1. Suppose that $p_2 \in H_{\epsilon_2} \subset U_2$ where H_{ϵ_2} is the horocycle of length $0 < \epsilon_2 \leq 2$ within the cusp region U_2.

Consider the peripheral monodromies ϕ_1 and ϕ_2 associated to κ. Therefore, some power $\phi_1^{\mu_1}$ is a multi-twist along a multi-curve $\boldsymbol{\alpha}_1$ and some power $\phi_2^{\mu_2}$ is a multi-twist along a multi-curve $\boldsymbol{\alpha}_2$, where both μ_1 and μ_2 are bounded above by a constant determined by h. There exist $\alpha_1 \in \boldsymbol{\alpha}_1$ and $\alpha_2 \in \boldsymbol{\alpha}_2$ such that $\iota(\alpha_1, \alpha_2) \geq 1$.

Take $q_1 = F(p_1)$, $q_2 = F(p_2) \in \mathcal{M}_h$ and set $\widetilde{q_1} = \widetilde{F}(\widetilde{p_1})$, $\widetilde{q_2} = \widetilde{F}(\widetilde{p_2}) \in \mathcal{T}_h$. By Proposition 4.6, since $d_{\mathcal{T}}(\widetilde{q_1}, \phi_1^{\mu_1} \cdot \widetilde{q_1}) \leq \mu_1 \cdot \epsilon_1 \leq 2\mu_1$ and $d_{\mathcal{T}}(\widetilde{q_2}, \phi_2^{\mu_2} \cdot \widetilde{q_2}) \leq \mu_2 \cdot \epsilon_2 \leq 2\mu_2$ we have

$$\begin{aligned} L_{\alpha_1}(\widetilde{q_1}) &\leq K_1(h, \mu_1) \cdot d_{\mathcal{T}}(\widetilde{q_1}, \phi_1^{\mu_1} \cdot \widetilde{q_1}) \leq K_1(h, \mu_1) \cdot \mu_1 \cdot \epsilon_1, \\ L_{\alpha_2}(\widetilde{q_2}) &\leq K_1(h, \mu_2) \cdot d_{\mathcal{T}}(\widetilde{q_2}, \phi_2^{\mu_2} \cdot \widetilde{q_2}) \leq K_1(h, \mu_2) \cdot \mu_2 \cdot \epsilon_2. \end{aligned}$$

Besides, since $\iota(\alpha_1, \alpha_2) \geq 1$, we have

$$\sinh\Big(\frac{L_{\alpha_1}(\widetilde{q_1})}{2}\Big) \sinh\Big(\frac{L_{\alpha_2}(\widetilde{q_1})}{2}\Big) \geq 1 \quad \text{and} \quad \sinh\Big(\frac{L_{\alpha_1}(\widetilde{q_2})}{2}\Big) \sinh\Big(\frac{L_{\alpha_2}(\widetilde{q_2})}{2}\Big) \geq 1$$

which implies that $L_{\alpha_1}(\widetilde{q_2}) \geq 2\,\mathrm{arcsinh}\,\frac{1}{\sinh K_1(h,\mu_2)\cdot\mu_2}$ and $L_{\alpha_2}(\widetilde{q_1}) \geq 2\,\mathrm{arcsinh}\,\frac{1}{\sinh K_1(h,\mu_1)\cdot\mu_1}$. Therefore, by Wolpert's Lemma, we have

$$\begin{aligned} d_{\mathbb{H}^2}(\widetilde{p_1},\widetilde{p_2}) &\geq 2d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_1}),\widetilde{F}(\widetilde{p_2})) = 2d_{\mathcal{T}}(\widetilde{q_1},\widetilde{q_2}) \geq \frac{1}{2}\log\frac{L_{\alpha_1(\widetilde{q_2})}}{L_{\alpha_1(\widetilde{q_1})}} + \frac{1}{2}\log\frac{L_{\alpha_2(\widetilde{q_1})}}{L_{\alpha_2(\widetilde{q_2})}} \\ &\geq \frac{1}{2}\big(\log\frac{2}{\epsilon_1} + \log\frac{2}{\epsilon_2}\big) - K_{5,1,2}(h,\mu_1,\mu_2) - K_{5,2,1}(h,\mu_1,\mu_2) \end{aligned}$$

where

$$\begin{aligned} K_{5,1,2}(h,\mu_1,\mu_2) &= \frac{1}{2}\log\frac{K_1(h,\mu_1)\cdot\mu_1}{\mathrm{arcsinh}\,\frac{1}{\sinh K_1(h,\mu_2)\cdot\mu_2}}, \\ K_{5,2,1}(h,\mu_1,\mu_2) &= \frac{1}{2}\log\frac{K_1(h,\mu_2)\cdot\mu_2}{\mathrm{arcsinh}\,\frac{1}{\sinh K_1(h,\mu_1)\cdot\mu_1}}. \end{aligned}$$

Using triangle inequality in $(\mathbb{H}^2, d_{\mathbb{H}^2})$ and the fact that D is convex and bounded by geodesic segments, we conclude that

$$d_{\mathbb{H}^2}(\widetilde{p_1},\widetilde{p_2}) \geq 2d_{\mathcal{T}}(\widetilde{q_1},\widetilde{q_2}) \geq \frac{1}{2}d_{\mathbb{H}^2}(\widetilde{p_1},\widetilde{p_2}) - \frac{1}{2}\mathrm{diam}(\widetilde{B_{cp}}\cap D) - K_{5,1,2}(h,\mu_1,\mu_2) - K_{5,2,1}(h,\mu_1,\mu_2)$$

where $\widetilde{B_{cp}} \subset \mathbb{H}^2$ is the lift of $B_{cp} \subset B$. □

5 Examples and applications

This section is intended to provide several examples, remarks concerning and consequences of Theorem A and Theorem B. We focus on holomorphic curves in $\mathcal{M}_2$. Let $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5$ and τ,σ be closed curves on Σ_2 represented in Figure 5 and Figure 6.

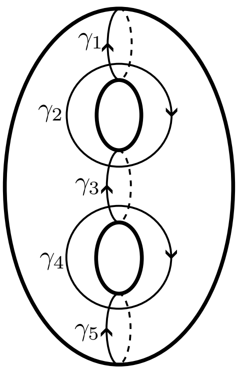


Figure 5: Five closed curves on Σ_2 along which the Dehn twists generate Mod_2. We have chosen their orientations for later use.

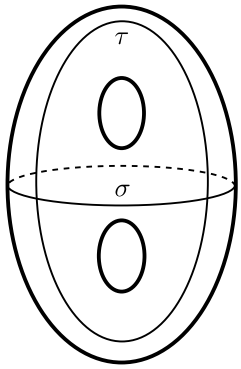


Figure 6: Another pair of closed curves on Σ_2.

It is well-known (cf. [Bir75, Theorem 4.8] and [Aur03, Figure 1]) that Mod_2 is generated by the five Dehn twists $T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5}$ and admits the following presentation:

$$\mathrm{Mod}_2 = \left\langle T_{\gamma_1},\dots,T_{\gamma_5} \,\middle|\, \begin{array}{c} T_{\gamma_i}\circ T_{\gamma_j} = T_{\gamma_j}\circ T_{\gamma_i} \quad \text{if } |i-j|>1; \\ T_{\gamma_i}\circ T_{\gamma_j}\circ T_{\gamma_i} = T_{\gamma_j}\circ T_{\gamma_i}\circ T_{\gamma_j} \quad \text{if } |i-j|=1; \\ T_{\gamma_1}\circ T_{\gamma_2}\circ T_{\gamma_3}\circ T_{\gamma_4}\circ T_{\gamma_5}^2\circ T_{\gamma_4}\circ T_{\gamma_3}\circ T_{\gamma_2}\circ T_{\gamma_1} = I \text{ is central;} \\ I^2 = 1; \quad (T_{\gamma_1}\circ T_{\gamma_2}\circ T_{\gamma_3}\circ T_{\gamma_4}\circ T_{\gamma_5})^6 = 1 \end{array} \right\rangle.$$

One can check that $I = (T_{\gamma_1}\circ T_{\gamma_2}\circ T_{\gamma_3}\circ T_{\gamma_4})^5$.

5.1 Quasi-isometrically but non-isometrically immersed curves. Theorem A provides a sufficient condition on the monodromy homomorphism for a holomorphic map $F: B \to \mathcal{M}_h$ to be a quasi-isometric immersion. On the other hand, the monodromy homomorphism of an isometric immersion $F: B \to \mathcal{M}_h$ is essentially purely pseudo-Anosov (see Definition 2.5 and Theorem 2.6). Therefore, we present a criterion for a holomorphic curve to be quasi-isometrically but not isometrically immersed.

Criterion 5.1. *Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Suppose that (a) the monodromy homomorphism is not essentially purely pseudo-Anosov, (b) all peripheral monodromies are of infinite order. Then, the holomorphic curve $F(B) \subset \mathcal{M}_h$ is quasi-isometrically but not isometrically immersed.*

In the rest of this subsection, we construct a quasi-isometrically but not isometrically immersed holomorphic curve $F(B) \subset \mathcal{M}_2$ of type $(0, 6)$. In addition, there exists a desired fundamental polygon D as in Theorem B such that $\left.\widetilde{F}\right|_D$ is a quasi-isometric embedding.

Example 5.2. Set $B = \mathbb{C} \setminus \{-2, -1, 0, 1, 2\}$ and

$$C' = \left\{ \big([X_0 : X_1], [Y_0 : Y_1], b\big) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \times B \,\middle|\, \begin{aligned} X_0^6 Y_1^2 =& (X_1 + X_0 b)(X_1 - X_0 b)(X_1 + X_0) \\ & (X_1 + 2X_0)(X_1 - X_0)(X_1 - 2X_0) Y_0^2 \end{aligned} \right\}.$$

Let $\pi' : C' \to B$ be a holomorphic map with $\pi'([X_0 : X_1], [Y_0 : Y_1], t) = t$. Then, each fibre $\pi'^{-1}(b)$ is a double cover of $\mathbb{C}P^1$ via $\pi'^{-1}(b) \ni ([X_0 : X_1], [Y_0 : Y_1], t) \mapsto [X_0 : X_1] \in \mathbb{C}P^1$ with branch points

$$P_1 = [1 : b], P_2 = [1 : 2], P_3 = [1 : 1], P_4 = [1 : -1], P_5 = [1 : -2], P_6 = [1 : -b] \text{ and } \infty = [0 : 1].$$

Therefore, the resolution at $[0 : 1]$ for every $t \in B$ is a holomorphic family C/B of Riemann surfaces of genus 2, say $\pi : C \to B$, which is non-isotrivial.

Proposition 5.3. *The classifying map of the holomorphic family C/B in Example 5.2 is a quasi-isometric but not-isometric immersion. Moreover, the lift of the classifying map restricted to some fundamental polygon is a quasi-isometric embedding.*

Proof. The base B is a Riemann surface of type $(0, 6)$. To illustrate the monodromy homomorphism, we fix the base point $t := 3 \in B$ and investigate generic fibres at $b \in \Gamma \subset B$ where Γ is shown in Figure 7. In fact, the resolution of $\pi'^{-1}(t)$ is the union of two copies of a single-valued branch that are glued along the boundary, where the boundary consists of three connected components. Figure 8 shows a piecewise correspondence between the algebraic curve $\pi^{-1}(t)$ and the topological surface Σ_2.

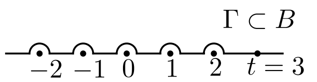


Figure 7: The subset $\Gamma \subset B$ at each point of which the fibre is generic.

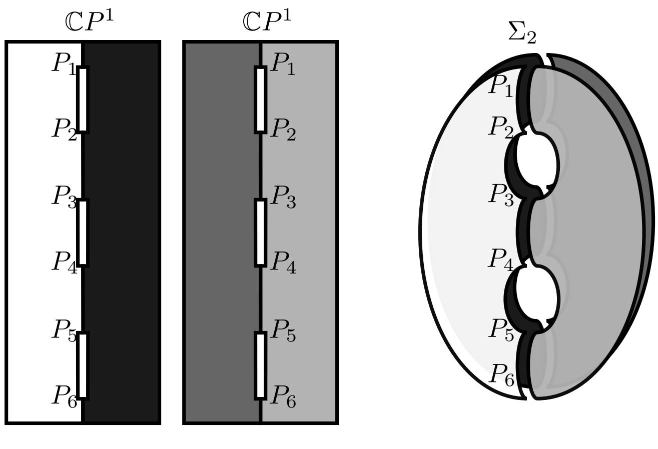


Figure 8: A piecewise correspondence between two copies of the single-valued branch at $t = 3$ and Σ_2.

When $b \in \Gamma$ is approaching one of -2, -1, 0, 1 and 2, there exist several closed curves on the generic fibre each joining two distinct branch points and vanishing when b takes the limit. These closed curves are called *vanishing cycles.* Figure 9 tells us what the pair of branch points is for each vanishing cycle. The peripheral monodromy is not the product of Dehn twists along vanishing cycles, but the product of their squares. One may compare its action on a transverse arc to the standard picture of a Dehn twist and a squared Dehn twist (see Figure 10).

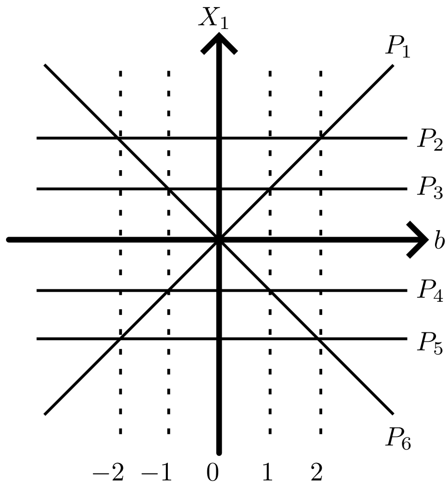


Figure 9: Deformation of the branch points.

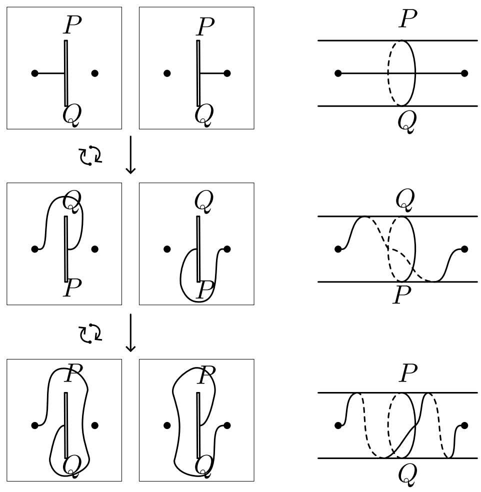


Figure 10: A pair of branch points rotated clockwise and the squared Dehn twist.

To identify the vanishing cycles, we deform the algebraic curve $\pi^{-1}(b)$ along Γ and maintain the correspondence with Σ_2. This deformation and all vanishing cycles at -2, -1, 0, 1, 2 are depicted in Figure 11.

We need the following.

Lemma 5.4. *Let $\gamma_1, \gamma_2 \subset \Sigma_2$ be two closed curves on the (oriented) closed surface of genus 2 such that $\iota(\gamma_1, \gamma_2) = 1$. Consider a path, denoted by $\gamma_1 \triangle \gamma_2$, starting from some $p \in \gamma_1 \setminus \gamma_2$, moving along γ_1 to the intersection, turning right, moving along γ_2 back to the intersection, turning left and moving along γ_1 back to p. Then, we have*

$$T_{\gamma_1 \triangle \gamma_2} = T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}.$$

Proof of Lemma 5.4. This comes from the fact that $\gamma_1 \triangle \gamma_2$ is homotopic to $T_{\gamma_2}^{-1}(\gamma_1)$. □

We return to the proof of Proposition 5.3. The monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_2$ is expressed as a sextuple, denoted by $(\phi_\infty, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2)$, where ϕ_∞, ϕ_{-2}, ϕ_{-1}, ϕ_0, ϕ_1 and ϕ_2 are peripheral monodromies at ∞, -2, -1, 0, 1 and 2. Using $\phi_2^{1/2}$, $\phi_1^{1/2}$, $\phi_0^{1/2}$, $\phi_{-1}^{1/2}$ and $\phi_{-2}^{1/2}$ to denote half peripheral monodromies at 2, 1, 0, -1 and -2 respectively, so that

$$\begin{aligned}
\phi_2^{1/2} =& T_{\gamma_1} \circ T_{\gamma_5},\\
\phi_1^{1/2} =& T_{\gamma_1 \triangle \gamma_2} \circ T_{\gamma_5 \triangle \gamma_4} = T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4},\\
\phi_0^{1/2} =& T_{(\gamma_1 \triangle \gamma_2) \triangle (\gamma_5 \triangle \gamma_4 \triangle \gamma_3)}\\
=& T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5}^{-1} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3},\\
\phi_{-1}^{1/2} =& T_{\gamma_1 \triangle \gamma_2 \triangle \gamma_3} \circ T_{\gamma_5 \triangle \gamma_4 \triangle \gamma_3} = T_{\gamma_3}^{-1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3},\\
\phi_{-2}^{1/2} =& T_{\gamma_1 \triangle \gamma_2 \triangle \gamma_3 \triangle \gamma_4} \circ T_{\gamma_4 \triangle \gamma_3 \triangle \gamma_2 \triangle \gamma_1}\\
=& T_{\gamma_4}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2},
\end{aligned}$$

we observe that

$$T_\tau^2 \circ (\phi_{-2}^{1/2})^2 \circ (\phi_{-1}^{1/2})^2 \circ (\phi_0^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ (\phi_2^{1/2})^2 = 1$$

and

$$(\phi_2^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ (\phi_2^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ T_{\gamma_2}^4 \circ T_{\gamma_4}^4 \circ T_\sigma^{-2} = 1$$

where τ and σ are given in Figure 6. Therefore, the peripheral monodromy at ∞ is again a multi-twist and there exists an essential closed curve on which the monodromy is a multi-twist. Hence, the holomorphic family C/B induces a quasi-isometrically but not isometrically immersed holomorphic curve.

The global monodromy $(\phi_\infty, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2)$ is a tuple in Mod_2 whose components are of infinite order and pairwise intersecting. Hence, the lift of the classifying map restricted to some fundamental polygon, say $\widetilde{F}\Big|_D : D \to \mathcal{T}_2$, is a quasi-isometric embedding. □

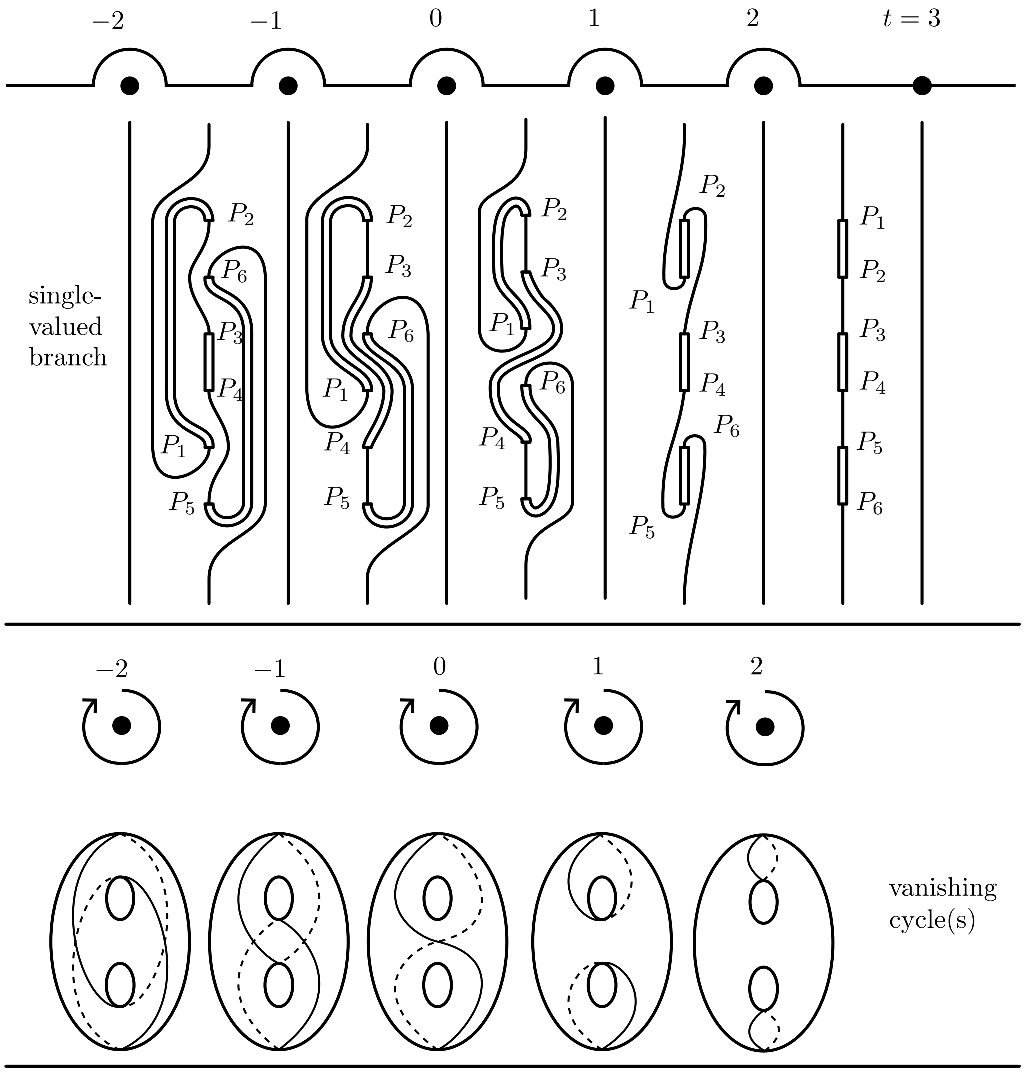


Figure 11: Deformation of the generic fibre along Γ is illustrated by the deformation of the single-valued branch, due to the correspondence between two copies of single-valued branch and Σ_2. One can further point out the vanishing cycle(s) at 2, 1, 0, −1 and −2 respectively.

5.2 Non quasi-isometrically embedded cusp regions. In this subsection, we provide a holomorphic curve of type $(0, 8)$ in $\mathcal{M}_2$ for which a cusp region is not quasi-isometrically embedded. In fact, the corresponding peripheral monodromy is of finite order and therefore this holomorphic curve does not satisfy the hypothesis of Theorem A - (i).

Example 5.5. Set $B = \mathbb{C} \setminus \{-1, -1/2, -1/3, 0, 1/3, 1/2, 1\}$ and

$$C = \left\{ \big([X_0 : X_1], [Y_0 : Y_1], b\big) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \times B \left| \begin{aligned} X_0^5 Y_1^2 = &(X_1 - (3b+2)X_0)(X_1 - bX_0) \\ &(X_1 + (3b-2)X_0)(X_1 + bX_0) \\ &(X_1 - X_0) Y_0^2 \end{aligned} \right. \right\}.$$

Let $\pi : C \to B$ be a holomorphic map with $\pi([X_0 : X_1], [Y_0 : Y_1], t) = t$. Then, each fibre $\pi^{-1}(b)$ is a double cover of $\mathbb{C}P^1$ via $\pi^{-1}(b) \ni ([X_0 : X_1], [Y_0 : Y_1], t) \mapsto [X_0 : X_1] \in \mathbb{C}P^1$ with branch points

$$P_1 = [1 : 3b+2], P_2 = [1 : b], P_3 = [1 : 1], P_4 = [1 : -b], P_5 = [1 : -3b+2] \text{ and } \infty = [0 : 1].$$

Therefore, C/B is a holomorphic family of Riemann surfaces of genus 2, say $\pi : C \to B$, which is non-isotrivial.

Proposition 5.6. *The classifying map $F : B \to \mathcal{M}_2$ of the holomorphic family C/B in Example 5.5 satisfies the following properties:*

- *Peripheral monodromies at* ∞ *are of order* 2.
- *The restriction of* F *to the cusp region at* ∞ *lies in a thick part of* $\mathcal{M}_2$.

Proof. The base B is a Riemann surface of type $(0, 8)$. To illustrate the monodromy homomorphism, we fix the base point $t := 2 \in B$ and investigate generic fibres at $b \in \Gamma \subset B$ where Γ is given in Figure 12.

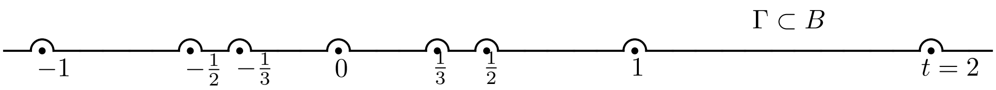


Figure 12: The subset $\Gamma \subset B$ of generic positions we are looking at.

Figure 13 shows a piecewise correspondence between $C_t := \pi^{-1}(t)$ and the topological surface Σ_2. Therefore, the monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_2$ is expressed as an octuple, denoted by $(\phi_\infty, \phi_{-1}, \phi_{-1/2}, \phi_{-1/3}, \phi_0, \phi_{1/3}, \phi_{1/2}, \phi_1)$, where ϕ_∞, ϕ_{-1}, $\phi_{-1/2}$, $\phi_{-1/3}$, ϕ_0, $\phi_{1/3}$, $\phi_{1/2}$ and ϕ_1 are peripheral monodromies at ∞, -1, $-1/2$, $-1/3$, 0, 1/3, 1/2 and 1.

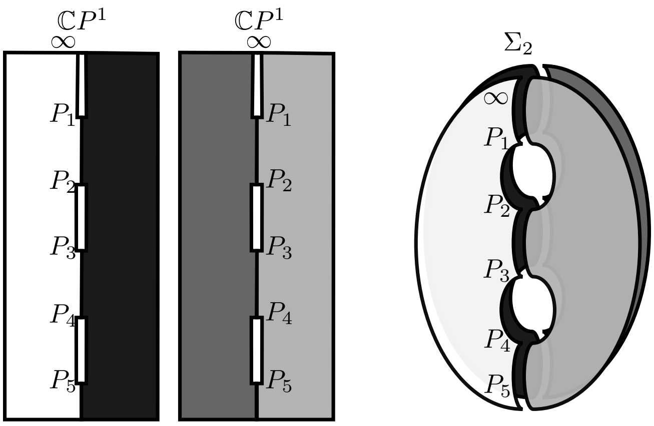


Figure 13: A piecewise correspondence between C_t and Σ_2.

The half peripheral monodromies are given by the following.

$$\begin{aligned}
\phi_1^{1/2} &= T_{\gamma_3} \circ T_{\gamma_5}, \\
\phi_{1/2}^{1/2} &= T_{\gamma_3 \triangle (\gamma_5 \triangle \gamma_4)} = T_{\gamma_4}^{-1} \circ T_{\gamma_5}^{-1} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4}, \\
\phi_{1/3}^{1/3} &= T_{\gamma_5 \triangle \gamma_4} = T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4}, \\
\phi_0^{1/2} &= T_{\gamma_3 \triangle \gamma_4} \circ T_{(\gamma_5 \triangle \gamma_4) \triangle (\gamma_3 \triangle \gamma_2)} \\
&= T_{\gamma_4}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_2} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_2}, \\
\phi_{-1/3}^{1/2} &= T_{\gamma_3 \triangle \gamma_2} = T_{\gamma_2}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_2}, \\
\phi_{-1/2}^{1/2} &= T_{(\gamma_3 \triangle \gamma_4) \triangle \gamma_2)} = T_{\gamma_2}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}, \\
\phi_{-1}^{1/2} &= T_{\gamma_2} \circ T_{\gamma_4}.
\end{aligned}$$

We observe that $(\phi_{-1}^{1/2})^2 \circ (\phi_{-1/2}^{1/2})^2 \circ (\phi_{-1/3}^{1/2})^2 \circ (\phi_0^{1/2})^2 \circ (\phi_{1/3}^{1/2})^2 \circ (\phi_{1/2}^{1/2})^2 \circ (\phi_1^{1/2})^2$ is of order 2.

Fix an orientation preserving diffeomorphism $f_t : \Sigma_2 \to C_t$ marking C_t and endow each $C_b := \pi^{-1}(b)$ with the marking f_b along Γ, for $b \in \Gamma$. We take a sufficiently large $N > 0$. To see that the restriction of F to the cusp region at ∞ lies in a thick part of $\mathcal{M}_2$, it suffices to show that $\mathrm{sys}(C_b)$ is bounded away from 0, for $b \in \mathbb{R}_{\geq N}$. From now on, we consider only generic positions $b \in \mathbb{R}_{\geq N}$ and generic fibres C_b at $b \in \mathbb{R}_{\geq N}$.

Half the hyperbolic distance on C_b is equal to the Kobayashi distance on C_b. Recall that the Kobayashi pseudo-norm on TC_b is defined by $\mathrm{Kob}_{C_b}(x, v) = \inf_\phi\{1/c\}$ for $x \in C_b$ and $v \in T_x C_b$, where the infimum is taken over all holomorphic maps $\phi : \Delta \to C_b$ satisfying $\phi(0) = x$ and $(d\phi)_0(\partial/\partial z) = c \cdot v$. In order to obtain a very coarse estimation of Kob_{C_b}, we make the following restrictions on ϕ: (i) $\phi(\Delta)$ lies in a single-valued branch, which is a subset of the affine chart $\mathbb{C}$; (ii) $\phi : \Delta \to \mathbb{C}$ is the composition of a linear map and a translation, meaning that $\phi(\Delta) \subset \mathbb{C}$ is also a disc away from $P_1, \ldots, P_5$.

The branch points $P_1, \ldots, P_5$ are colinear. We observe that each ratio

$$r_{i,j,k,l}(b) \coloneqq \frac{d_{\mathbb{R}^2}(P_i, P_j)}{d_{\mathbb{R}^2}(P_k, P_l)}$$

of euclidean distances converges as $b \to \infty$, for $i \neq j$ and $k \neq l$. Set

$$r_{\min} \coloneqq (1/2) \min_{i,j,k,l} \{ \lim_{b\to\infty} r_{i,j,k,l}(b) \} \quad \text{and} \quad r_{\max} \coloneqq (1/2) \max_{i,j,k,l} \{ \lim_{b\to\infty} r_{i,j,k,l}(b) \}.$$

Therefore, for each $i \in \{2, 3, 4, 5\}$, there exists a closed curve β_i lying in a single-valued branch and homotopic to $f_b(\gamma_i)$ such that $l_{\mathbb{R}^2}(\beta_i) \leq 12r_{\max} + \epsilon$ and $d_{\mathbb{R}^2}(x, P_j) \geq r_{\min} - \epsilon$ for all $x \in \beta_i$ and $j \in \{1, 2, 3, 4, 5\}$, where $\epsilon > 0$ is sufficiently small and determined by N (see Figure 14).

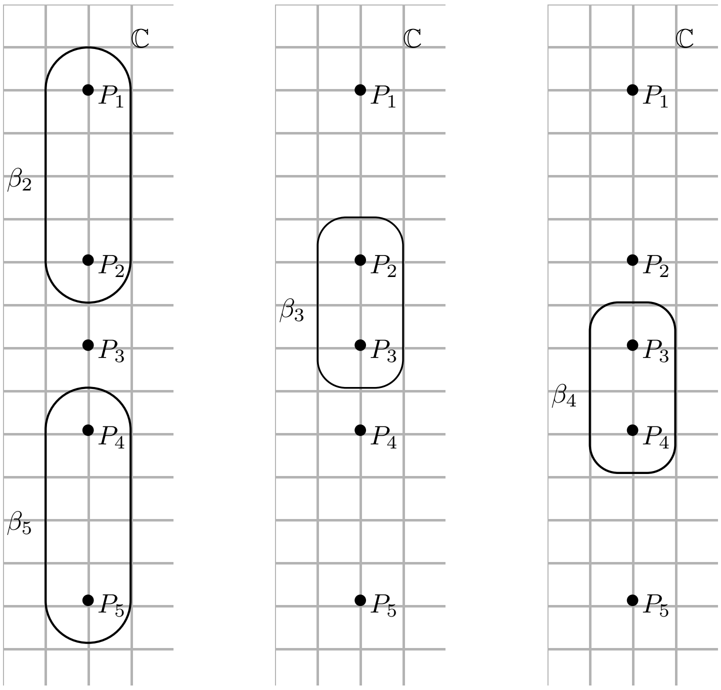


Figure 14: Closed curves homotopic to $f_b(\gamma_2), \ldots, f_b(\gamma_5)$ with bounded Kobayashi lengths.

Each of β_2, β_3, β_4 and β_5 has the hyperbolic length

$$\begin{aligned} l_{C_b}(\beta_i) &= 2 \cdot \int_0^1 \mathrm{Kob}_{C_b}(\beta_i(t), \dot\beta_i(t))dt \leq 2 \cdot \int_0^1 \frac{1}{\min_j \{ d_{\mathbb{R}^2}(\beta_i(t), P_j) \} / |\dot\beta_i(t)|} dt \\ &\leq \frac{2}{r_{\min} - \epsilon} \int_0^1 |\dot\beta_i(t)| dt = \frac{2 l_{\mathbb{R}^2}(\beta_i)}{r_{\min} - \epsilon} \leq \frac{2(12 r_{\max} + \epsilon)}{r_{\min} - \epsilon}. \end{aligned}$$

Hence $L_{\gamma_2}(C_b)$, $L_{\gamma_3}(C_b)$, $L_{\gamma_4}(C_b)$ and $L_{\gamma_5}(C_b)$ are uniformly bounded from above. We conclude that $\mathrm{sys}(C_b)$ is bounded away from 0. □

5.3 Holomorphic genus-2 Lefschetz fibrations. Holomorphic genus-2 Lefschetz fibrations over a punctured sphere are quite well-understood by works of Siebert and Tian [ST05] as well as Chakiris [Cha83] and Smith [Smi99] (cf. also [Sal14]). In particular, there are only finitely many explicit possibilities for the global monodromies of genus-2 Lefschetz fibrations without reducible fibres (i.e. without separating vanishing cycles) up to Hurwitz moves. By convention, we use $\bullet$ to denote the concatenation of tuples: $(\phi_1, \ldots, \phi_k) \bullet (\psi_1, \ldots, \psi_l) = (\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_l)$. The power of a tuple corresponds to a repeated concatenation with itself. The symbol $\prod$ represents the concatenation of a family of tuples.

Proposition 5.7. *Given $n \geq 3$, let B be an oriented hyperbolic surface of type $(0, n)$ and $F : B \to \mathcal{M}_2$ be a holomorphic map with a global monodromy $(\phi_1, \ldots, \phi_n)$. Suppose each ϕ_i is the Dehn twist along a non-separating closed curve. Then the global monodromy has the following properties:*

(i) Using a finite sequence of Hurwitz moves, one can transform $(\phi_1, \ldots, \phi_n)$ into the concatenation of tuples $\mathcal{A}_1^p \bullet \mathcal{A}_2^q \bullet \mathcal{A}_3^r$ where p, q, r are non-negative integers and

$$\mathcal{A}_1 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5}, T_{\gamma_5}, T_{\gamma_4}, T_{\gamma_3}, T_{\gamma_2}, T_{\gamma_1})^2,$$
$$\mathcal{A}_2 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4})^5, \ \mathcal{A}_3 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5})^6.$$

(ii) Using a finite sequence of Hurwitz moves, one can transform $(\phi_1, \ldots, \ldots_n)$ into a tuple of Dehn twists along pairwise intersecting closed curves.

Proposition 5.7 - (i) is Theorem B in [Cha83] and the holomorphic map $F : B \to \mathcal{M}_2$ is the classifying map of a holomorphic genus-2 Lefschetz fibration without separating vanishing cycles. In this case, by Proposition 5.7 - (ii), all hypotheses of Theorem B hold for a specific fundamental polygon D and we have the following corollary.

Corollary 5.8. *Let $B = \Gamma\backslash\mathbb{H}^2$ be an oriented hyperbolic surface of type $(0, n)$, $n \geq 3$. Let $F : B \to \mathcal{M}_2$ be the classifying map of a holomorphic genus-2 Lefschetz fibration without separating vanishing cycles. Then, there exists a fundamental polygon D of B such that $\widetilde{F}\Big|_D : (D, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_2, d_{\mathcal{T}})$ is a $(2, K + \mathrm{diam}(D))$-quasi-isometric embedding, where $K = K(0, n, 2, \mathrm{sys}(B))$ as in Theorem B.*

For positive integer l, recall that Hurwitz moves acting on a l-tuple $(\phi_1, \ldots, \phi_k)$ are given by

$$(\ldots, \phi_i \circ \phi_{i+1} \circ \phi_i^{-1}, \phi_i, \ldots) \xleftarrow{L_i} (\ldots, \phi_i, \phi_{i+1}, \ldots) \xrightarrow{R_i} (\ldots, \phi_{i+1}, \phi_{i+1}^{-1} \circ \phi_i \circ \phi_{i+1}, \ldots).$$

We consider a closed curve $\delta \subset \Sigma_2$ and use $\overrightarrow{\delta}$ to denote an orientation of δ. The *algebraic intersection number* of two oriented closed curves $\overrightarrow{\delta_1}$ and $\overrightarrow{\delta_2}$, denoted by $\hat{\imath}(\overrightarrow{\delta_1}, \overrightarrow{\delta_2})$, is defined as the sum of the indices of the intersection points of $\overrightarrow{\delta_1}$ and $\overrightarrow{\delta_2}$, where an intersection point is of index $+1$ when the orientation of the intersection agrees with the orientation of $\Sigma_{g,n}$ and is -1 otherwise. Note that $\hat{\imath}(\overrightarrow{\delta_1}, \overrightarrow{\delta_2}) \neq 0$ only if $\iota(\delta_1, \delta_2) \neq 0$.

Let Ω_l be the set of l-tuples $(g_1, \ldots, g_l)$ where each g_i is a positive Dehn twist in Mod_2, ${}^\sharp\Omega_l$ be the set of l-tuples $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l})$ where each $\overrightarrow{\delta_i}$ is an orientation of some closed curve $\delta_i \subset \Sigma_2$. There is a natural map $\natural : {}^\sharp\Omega_l \to \Omega_l$ sending $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l})$ to $(T_{\delta_1}, \ldots, T_{\delta_l})$. We define the *matrix of algebraic intersections* $\hat{M} = \hat{M}(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) \in \mathbb{R}^{l\times l}$ on every element in ${}^\sharp\Omega_l$ by setting $\hat{M}_{i,j} = \hat{\imath}(\overrightarrow{\delta_i}, \overrightarrow{\delta_j})$.

The maps ${}^\sharp L_i$ and ${}^\sharp R_i$ on ${}^\sharp\Omega_l$ induced by the Hurwitz moves L_i and R_i are defined as follows.

$$(\ldots, T_{\delta_i}(\overrightarrow{\delta_{i+1}}), \overrightarrow{\delta_i}, \ldots) \xleftarrow{{}^\sharp L_i} (\ldots, \overrightarrow{\delta_i}, \overrightarrow{\delta_{i+1}}, \ldots) \xrightarrow{{}^\sharp R_i} (\ldots, \overrightarrow{\delta_{i+1}}, T_{\delta_{i+1}}^{-1}(\overrightarrow{\delta_i}), \ldots).$$

We also have the maps ${}^\flat L_i$ and ${}^\flat R_i$ on $\mathbb{R}^{l\times l}$ defined by

$$\hat{M}' = (m'_{j,k}) \xleftarrow{{}^\flat L_i} \hat{M} = (m_{j,k}) \xrightarrow{{}^\flat R_i} \hat{M}'' = (m''_{j,k})$$

such that $m'_{i,i} = m'_{i+1,i+1} = m''_{i,i} = m''_{i+1,i+1} = 0$, $m'_{i,i+1} = m_{i+1,i} = m''_{i,i+1}$, $m'_{i+1,i} = m_{i,i+1} = m''_{i+1,i}$ and, for $j, k \notin \{i, i+1\}$, that $m'_{j,k} = m_{j,k} = m''_{j,k}$,

$$m'_{i,k} = m_{i+1,k} + m_{i+1,i}m_{i,k}, \ m'_{i+1,k} = m_{i,k} \ , m'_{j,i} = m_{j,i+1} - m_{i,i+1}m_{j,i}, \ m'_{j,i+1} = m_{j,i},$$
$$m''_{i,k} = m_{i+1,k}, \ m''_{i+1,k} = m_{i,k} - m_{i,i+1}m_{i+1,k}, \ m''_{j,i} = m_{j,i+1}, \ m''_{j,i+1} = m_{j,i} + m_{i+1,i}m_{j,i+1}.$$

Suppose that q is a sequence of Hurwitz moves. We use ${}^\sharp q$ to denote the sequence of corresponding maps on ${}^\sharp\Omega_l$ and use ${}^\flat q$ to denote the sequence of corresponding maps on $\mathbb{R}^{l\times l}$.

Proposition 5.9. *Let $(g_1, \ldots, g_l) \in \Omega_l$ be a tuple of positive Dehn twists in Mod_2 where $g_i = T_{\delta_i}$ for $i = 1, \ldots, l$. Let q be a sequence of Hurwitz moves. Suppose that $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) \in {}^\sharp\Omega_l$ is a lift of $(g_1, \ldots, g_l)$ and $(\overrightarrow{\delta'_1}, \ldots, \overrightarrow{\delta'_l}) \in {}^\sharp\Omega_l$ is a lift of the resulting tuple $q \cdot (g_1, \ldots, g_l)$. Then*

$${}^\flat q \cdot \hat{M}(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) = \hat{M}(\overrightarrow{\delta'_1}, \ldots, \overrightarrow{\delta'_l}).$$

Proof. It suffices to show that the following diagram is commutative.

$$\begin{array}{ccccc}
\Omega_l & \xleftarrow{L_i} & \Omega_l & \xrightarrow{R_i} & \Omega_l \\
\natural\big\uparrow & & \natural\big\uparrow & & \natural\big\uparrow \\
{}^\sharp\Omega_l & \xleftarrow{{}^\sharp L_i} & {}^\sharp\Omega_l & \xrightarrow{{}^\sharp R_i} & {}^\sharp\Omega_l \\
\hat{M}\big\downarrow & & \hat{M}\big\downarrow & & \hat{M}\big\downarrow \\
\mathbb{R}^{l\times l} & \xleftarrow{{}^\flat L_i} & \mathbb{R}^{l\times l} & \xrightarrow{{}^\flat R_i} & \mathbb{R}^{l\times l}
\end{array}$$

On the one hand, we have $T_{T_{\delta_i}(\delta_{i+1})} = T_{\delta_i} \circ T_{\delta_{i+1}} \circ T_{\delta_i}^{-1}$ and $T_{T^{-1}_{\delta_{i+1}}(\delta_i)} = T^{-1}_{\delta_{i+1}} \circ T_{\delta_i} \circ T_{\delta_{i+1}}$ (see, e.g., [FM11, Fact 3.7]). On the other hand, the algebraic intersection is well-defined on homology classes and we have

$$[T_{\delta_i}(\overrightarrow{\delta_{i+1}})] = [\overrightarrow{\delta_{i+1}}] + \hat{\imath}(\overrightarrow{\delta_{i+1}}, \overrightarrow{\delta_i}) \cdot [\overrightarrow{\delta_i}], \quad [T^{-1}_{\delta_{i+1}}(\overrightarrow{\delta_i})] = [\overrightarrow{\delta_i}] - \hat{\imath}(\overrightarrow{\delta_i}, \overrightarrow{\delta_{i+1}}) \cdot [\overrightarrow{\delta_{i+1}}]$$

for $i = 1, \dots, l-1$ (see, e.g., [FM11, Proposition 6.3]). □

We prove Proposition 5.7 - (ii) by starting with Lemma 5.10 with a computer-assisted proof.

Lemma 5.10. *For $i = 1, 2, 3$, there exists a finite sequence of Hurwitz moves q_i that satisfies the following statement. Suppose that $\mathcal{A}_i$ is transformed by q_i into a tuple of Dehn twists, denoted by $(T_{\delta_{i,1}}, \dots, T_{\delta_{i,l_i}})$. Then the algebraic intersection between each two of $\delta_{i,1}, \dots, \delta_{i,l_i}$ and γ_1 is non-zero.*

Proof. We consider the sequence of Hurwitz moves q_i below and show that the resulting l_i-tuple $(T_{\delta_{i,1}}, \dots, T_{\delta_{i,l_i}}) = q_i \cdot \mathcal{A}_i$ satisfies the desired properties. Applying q_i to the (l_i+1)-tuple $\mathcal{A}_i \bullet \{T_{\gamma_1}\}$, we obtain a (l_i+1)-tuple of the form $(T_{\delta_{i,1}}, \dots, T_{\delta_{i,l_i}}, T_{\gamma_1})$ since each component in q_i is neither L_{l_i} nor R_{l_i}. Suppose that $(\overrightarrow{\delta_1}, \dots, \overrightarrow{\delta_{l_i+1}}) \in {}^\sharp\Omega_{l_i}$ is the lift of $\mathcal{A}_i \bullet \{T_{\gamma_1}\}$ where the orientations of $\overrightarrow{\gamma_1}, \dots, \overrightarrow{\gamma_5}$ are shown in Figure 5. Consider the matrix of algebraic intersections $\hat{M} = \hat{M}(\overrightarrow{\delta_1}, \dots, \overrightarrow{\delta_{l_i+1}})$ and apply ${}^\flat q_i$ to $\hat{M}$. By Proposition 5.9, it suffices to verify that ${}^\flat q_i \cdot \hat{M}$ has non-zero off-diagonal entries.

$$\begin{aligned}
q_1 = (&R_3, L_{10}, L_2, R_{16}, L_{17}, L_1, R_8, R_{16}, R_{11}, R_{12}, R_4, L_{13}, L_{11}, L_{15}, R_{10}, R_{13}, L_5, R_{12}, R_9, L_8,\\
&R_2, L_6, L_{18}, L_7, R_6, R_{19}, R_4, L_8, R_5, L_9, L_9, L_{14}, L_8, L_8, R_{18}, L_{10}, R_{13}, L_6, L_6, R_{19},\\
&R_7, R_{12}, L_8, R_7, L_4, L_{11}, R_3, R_2, R_{12}, R_2, R_{13}, L_{10}, L_5, L_4, L_{10}, L_{17}, L_{18}, R_{11}, L_{12}, L_{13},\\
&R_6, R_4, L_{10}, L_{12}, L_{13}, L_{14}, R_4, R_{15}, R_1, L_{12}, L_{16}, L_{17}, R_{16}, L_{18}, L_{14}, L_{18}, R_{17}, R_1, L_{14}, L_{14},\\
&R_3, L_5, L_5),\\
q_2 = (&R_8, R_{15}, L_2, L_6, R_{10}, L_{13}, R_{16}, R_5, L_{17}, L_8, R_{14}, R_3, R_{11}, L_{10}, R_6, L_{12}, R_8, L_4, L_7, L_9,\\
&R_3, R_6, R_{14}, L_{12}, R_{11}, L_9, R_{16}, L_1, L_2, L_{17}, R_7, R_{13}, L_8, R_{14}, R_{14}, R_1, L_{15}, L_9, L_{18}, L_{14},\\
&R_{10}, R_{13}, L_1, R_{16}, L_{15}, R_1, R_7, R_{10}, R_{10}, L_{19}, R_{16}, R_{15}, R_3),\\
q_3 = (&L_{12}, R_{18}, L_7, R_{24}, L_{14}, R_7, L_{10}, L_{27}, L_5, R_1, R_{19}, R_{10}, L_{24}, R_{20}, L_{28}, L_{27}, L_{11}, L_{22}, L_8, L_{10},\\
&R_9, R_{24}, R_7, R_2, R_{19}, L_{10}, R_{25}, R_{26}, L_{25}, L_8, R_1, R_{10}, R_{20}, L_{21}, L_{20}, L_{25}, L_3, L_{10}, R_{23}, L_9,\\
&L_7, R_6, R_{10}, L_{13}, L_8, L_{27}, R_{24}, L_{21}, R_{22}, R_{12}, R_6, L_{11}, L_5, R_{14}, R_8, L_{11}, L_{18}, R_7, R_4, L_{12},\\
&R_{15}, L_{20}, R_7, L_{11}, L_{12}, R_{10}, L_{19}, L_{10}, R_3, R_{17}, L_{14}, L_{23}, L_9, L_{25}, R_{24}, R_4, R_{22}, L_{13}, L_{18}, R_3,\\
&L_{29}, L_{24}, R_{22}, R_{22}, L_{16}, L_{18}, R_{28}, L_{29}, L_{29}, L_6, R_1, L_{24}, L_{14}, R_1, L_{20}, R_{17}, L_8, R_{25}, R_{12}, R_{14},\\
&R_8, L_1, L_{20}, L_{22}, R_7, L_{18}, R_{14}, L_{21}, R_7, R_{18}, R_9, R_{14}, L_2, R_1, L_{19}, R_{11}, R_{27}, R_9, R_4, L_{26},\\
&R_{20}, L_{10}, L_{23}, L_{22}, R_{20}, R_{17}, R_5, R_5, L_8).
\end{aligned}$$

□

In the proof of Lemma 5.10 the matrix ${}^\flat q_i \cdot \hat{M}$ is hard to obtain manually but can be quickly solved by a computer. We implement these computations in Python and make our code available on GitHub: `https://github.com/AHdoc/HurwitzMoves_to_AlgIntersections`.

Proof of Theorem 5.7 - (ii). By Proposition 5.7 - (i) and Lemma 5.10, using a sequence of Hurwitz moves, one can always transform a global monodromy into the concatenation of sub-tuples

$$\mathcal{A}_u \bullet (T_{\delta_{1,1}}, \dots, T_{\delta_{1,l_1}})^{p'-1} \bullet (T_{\delta_{2,1}}, \dots, T_{\delta_{2,l_2}})^{q'-1} \bullet (T_{\delta_{3,1}}, \dots, T_{\delta_{3,l_3}})^{r'-1}$$

with some $u \in \{1, 2, 3\}$ and non-negative integers p', q', r' such that $p' + q' + r' - 3 = p + q + r - 1$, $\hat{\imath}(\gamma_1, \delta_{i,j}) \neq 0$ for $i = 1, 2, 3$, $j = 1, \dots, l_i$ and $\hat{\imath}(\delta_{i,j}, \delta_{i,k}) \neq 0$ for $i = 1, 2, 3$, $1 \leq j \neq k \leq l_i$. By Lemma 4.2, making N sufficiently large such that $N - 2 > \iota(\delta_{i,j}, \delta_{i',j'})$ for all $i = 1, 2, 3$, $i' = 1, 2, 3$, $1 \leq j \leq l_i$, $1 \leq j' \leq l_{i'}$, we have

$$\iota(T^N_{\gamma_1}(\delta_{i,j}), \delta_{i',j'}) \geq (N-2)\,\iota(\gamma_1, \delta_{i,j})\,\iota(\gamma_1, \delta_{i',j'}) - \iota(\delta_{i,j}, \delta_{i',j'}) \geq 1.$$

Since $T_{\delta_{i,1}} \cdots T_{\delta_{i,l_i}}$ is central for $i = 1, 2, 3$, one can further transform the global monodromy into

$$\mathcal{A}_u \bullet \prod_{j=1}^{p'} (T_{T_{\gamma_1}^{jN}(\delta_{1,1})}, \ldots, T_{T_{\gamma_1}^{jN}(\delta_{1,l_1})}) \bullet \prod_{j=p'+1}^{p'+q'} (T_{T_{\gamma_1}^{jN}(\delta_{2,1})}, \ldots, T_{T_{\gamma_1}^{jN}(\delta_{2,l_2})}) \bullet \prod_{j=p'+q'+1}^{p'+q'+r'} (T_{T_{\gamma_1}^{jN}(\delta_{3,1})}, \ldots, T_{T_{\gamma_1}^{jN}(\delta_{3,l_3})}).$$

Applying Lemma 5.10 again, we replace $\mathcal{A}_u$ with $(T_{\delta_{u,1}}, \ldots, T_{\delta_{u,l_u}})$, as desired. □

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