4-manifolds admitting fibrations on surfaces

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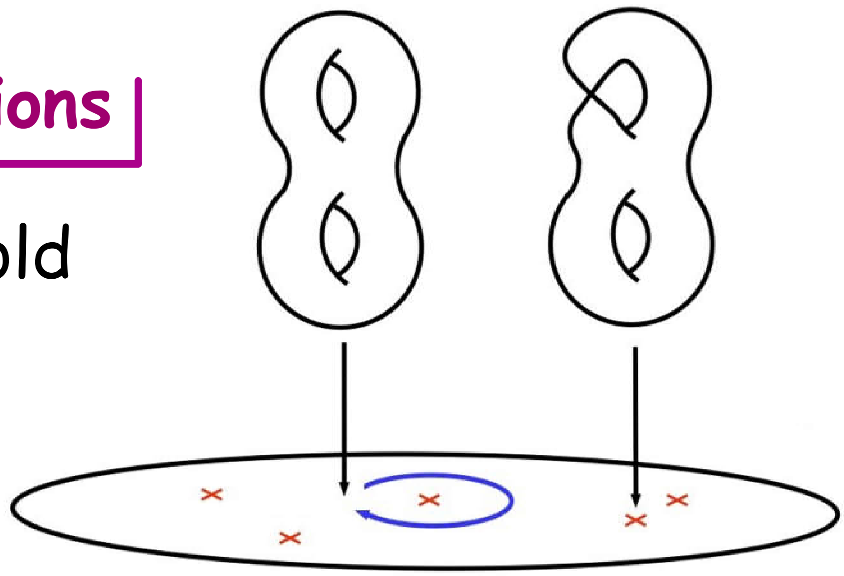
4-manifolds admitting fibrations

- M^4: closed oriented 4-manifold
- B : closed oriented surface
- $S=\{p_1,\dots,p_n\}\subset B$ branch set

4-manifolds admitting fibrations



- M^4: closed oriented 4-manifold
- B : closed oriented surface
- $S=\{p_1,\dots,p_n\}\subset B$ branch set

A genus-h fibration of M^4 over B branched over S is $f:M^4\to B$ s.t.

$$f|_{M\setminus f^{-1}(S)}: M\setminus f^{-1}(S) \to B\setminus S$$

is a locally trivial fibration (i.e. a fibre bundle)

& each generic fibre $f^{-1}(b)$ is a closed surface of genus h

Monodromy of $f : M^4 \to B$

(generic) base point $t \in B \setminus S$

⟶ monodromy homomorphism $\Phi_{f,t} : \pi_1\big(B\setminus S, t\big) \to \mathrm{Mod}\big(f^{-1}(t)\big)$

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$\rightarrow$ monodromy homomorphism $\Phi_{f,t} : \pi_1(B \setminus S, t) \to \mathrm{Mod}(f^{-1}(t))$

Mapping class groups $\mathrm{Mod}_{g,n} := \mathrm{Mod}(\Sigma_{g,n})$, $\mathrm{Mod}_h := \mathrm{Mod}(\Sigma_h)$.
Take homeomorphisms $\varphi : \Sigma_h \to f^{-1}(t)$ and $\psi : (\Sigma_{g,n}, s) \to (B \setminus S, t)$.

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Definition (monodromy invariant) :

MO(f) is the coset of $\varphi_*^{-1} \circ \Phi_{f,t} \circ \psi_*$ in

$$M_{g,n,h} := \mathrm{Mod}_{g,n} \backslash \mathrm{Hom}(\pi_1(\Sigma_{g,n}, s), \mathrm{Mod}_h) / \mathrm{Mod}_h .$$

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$\longrightarrow$ monodromy homomorphism $\Phi_{f,t} : \pi_1(B \setminus S, t) \to \mathrm{Mod}(f^{-1}(t))$

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Remarks:
- MO(f) does not depend on t, φ and ψ.
- $MO(f_1) = MO(f_2)$ **iff**

Page 2 $\exists$ fibre-preserving homeomorphism between $M_i \setminus f_i^{-1}(S_i)$

Study of genus-h fibration by mean of monodromy

Question 1: To what extent $MO(f)$ determines (M^4, f, B) ?

Question 2: How to describe $M_{g,n,h}$?

Study of genus-h fibration by mean of monodromy

Question 1: To what extent $MO(f)$ determines (M^4, f, B) ?

Question 2: How to describe $M_{g,n,h}$?

Today: Part I: torus fibration over 2-sphere

- classify elements of $M_{0,n,1}$ up to stabilisation

Part II: holomorphic fibration $f : C^2 \to B$

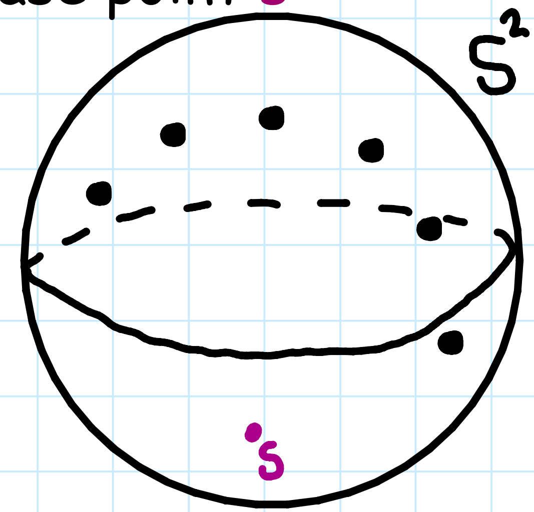
- Finiteness of $\{MO(f) \mid f : C^2 \to B$ holomorphic$\}$ $h \geq 2$
- Classifying map $F : B \to \mathcal{M}_h$

f: $M^4 \to S^2$
Generic fibre = torus
$\mathrm{Mod}_1 = SL(2,\mathbb{Z})$
base point **s**

Part I-(i) Torus fibration over S^2

Monodromy homomorphism:

$$\Phi_{f,s} : \pi_1(S^2 \setminus \{p_1, \dots, p_n\}, s) \longrightarrow SL(2,\mathbb{Z})$$



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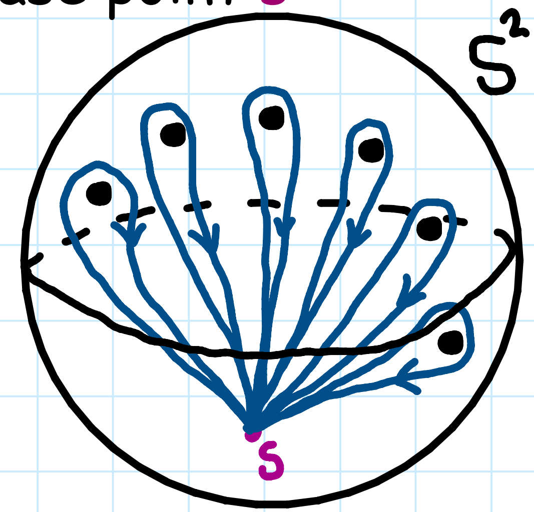

Choose generator loops $r_1, r_2, \ldots, r_n$ s.t.

$$\pi_1(S^2 \setminus \{p_1, \ldots, p_n\}, s) = \langle r_1, r_2, \ldots, r_n \mid r_1 \ldots r_n = 1 \rangle.$$

$\Phi_{f,s}$ is identified with

an n-tuple $(\phi_1, \ldots, \phi_n)$ global monodromy.

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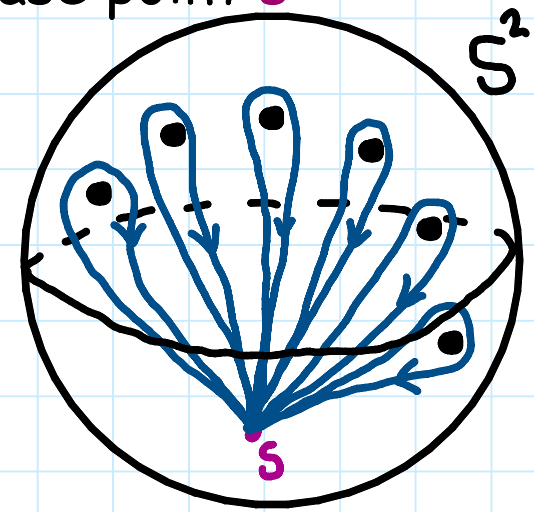
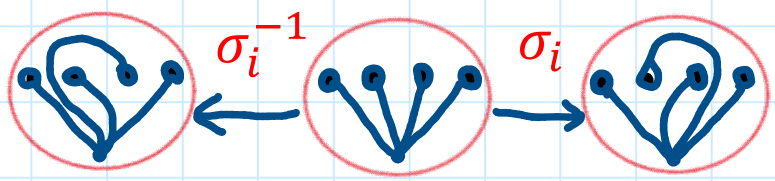


$B_n = B_n(\mathbb{D}^2) \twoheadrightarrow B_n(S^2) \twoheadrightarrow Mod_{0,n}$

Artin generators σ_i provide



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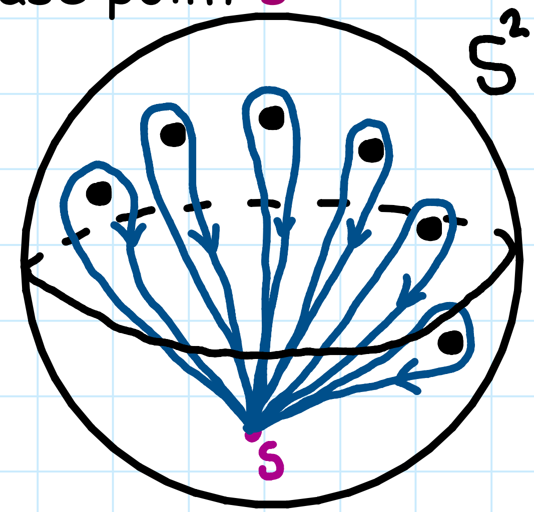

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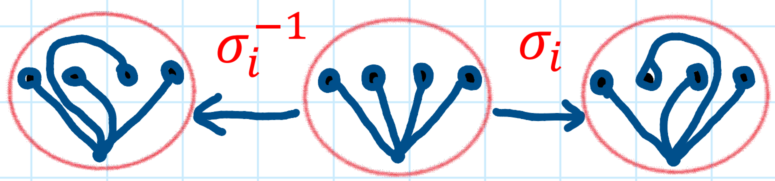

Hurwitz moves:

$$(\ldots, \phi_i \phi_{i+1} \phi_i^{-1}, \phi_i, \ldots) \xleftarrow{R_i^{-1}} (\ldots, \phi_i, \phi_{i+1}, \ldots) \xrightarrow{R_i} (\ldots, \phi_{i+1}, \phi_{i+1}^{-1} \phi_i \phi_{i+1}, \ldots)$$

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Choose generator loops $r_1, r_2, \dots, r_n$ s.t.

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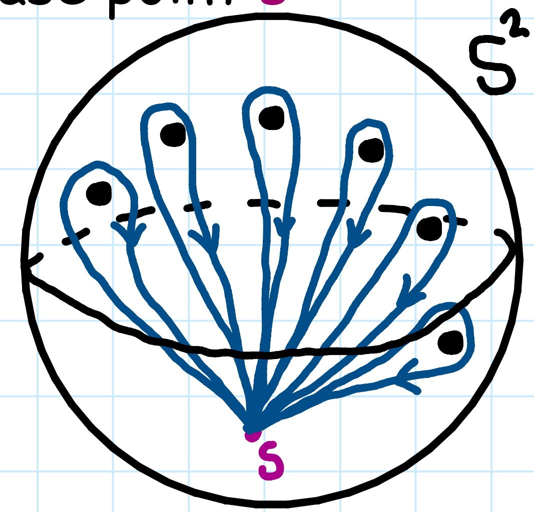

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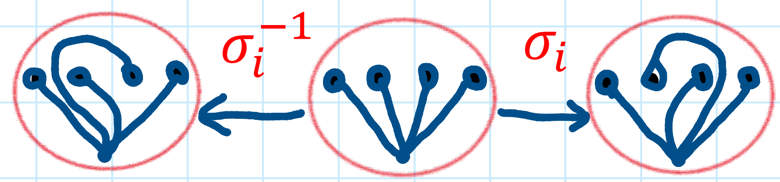

Hurwitz moves:

$$(\dots, \phi_i \phi_{i+1} \phi_i^{-1}, \phi_i, \dots) \xleftarrow{R_i^{-1}} (\dots, \phi_i, \phi_{i+1}, \dots) \xrightarrow{R_i} (\dots, \phi_{i+1}, \phi_{i+1}^{-1} \phi_i \phi_{i+1}, \dots)$$

Hurwitz equivalent: $(\phi_1, \dots, \phi_n) \sim (\psi_1, \dots, \psi_n)$

Claim: Hurwitz equivalence $\rightsquigarrow$ same invariant in $M_{0,n,1}$

Part I-(ii) Type of singularities

$f\colon M^4 \to S^2$ $(\phi_1,\dots,\phi_n)$

Definition: type $\mathcal{O}(f) := [\,[\phi_1], \dots, [\phi_n]\,]$ a multi-set.

Remark: $(\phi_1, \dots, \phi_n) \sim (\phi_1', \dots, \phi_n') \Longrightarrow MO(f) = MO(f') \Longrightarrow \mathcal{O}(f) = \mathcal{O}(f')$.

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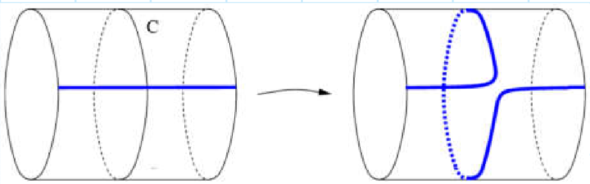
Examples:

(1) torus Lefschetz fibration :

- Local model for singularities: $f(z_1, z_2) = z_1^2 + z_2^2$ orientation-preserving chart
- Each ϕ_i is a positive Dehn twist
- $\mathcal{O}(f) = n \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = n \cdot I_1^+$



Thm. (Moishezon, Livné 1977) :

Let f and f' be torus Lefschetz fibrations over S^2. Then
$\mathcal{O}(f) = \mathcal{O}(f') \Leftrightarrow (\phi_1, \dots, \phi_n) \sim (\phi_1', \dots, \phi_n')$.

Part I-(ii) Type of singularities

$f: M^4 \to S^2 \quad (\phi_1, \dots, \phi_n)$

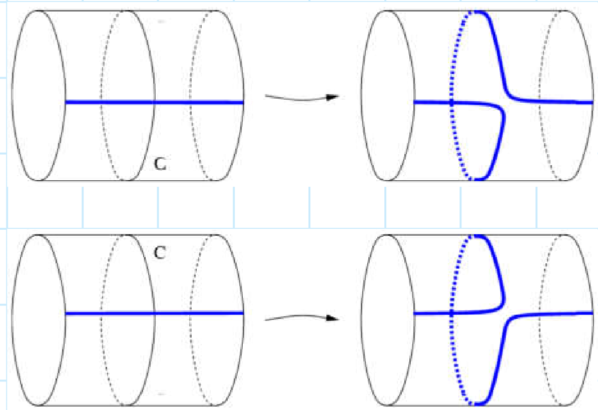
Definition: type $\mathcal{O}(f) := [\,[\phi_1], \dots, [\phi_n]\,]$ a multi-set.

Remark: $(\phi_1, \dots, \phi_n) \sim (\phi_1', \dots, \phi_n') \Rightarrow MO(f) = MO(f') \Rightarrow \mathcal{O}(f) = \mathcal{O}(f')$.

Examples:

(2) torus achiral Lefschetz fibration :

- Local model for singularities: $f(z_1, z_2) = z_1^2 + z_2^2$
- Each ϕ_i is a positive/negative Dehn twist
- $\mathcal{O}(f) = p \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + q \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = p \cdot I_1^+ + q \cdot I_1^-$



Thm. D (Matsumoto '85; Z.) :

Let f and f' be torus achiral Lefschetz fibrations over S^2 such that $\mathcal{O}(f) = \mathcal{O}(f') = p \cdot I_1^+ + q \cdot I_1^-$. Then

$p \neq q$: $(\phi_1, \dots, \phi_n) \sim (\phi_1', \dots, \phi_n')$;

$p = q \geq 1$: ∞ many Hurwitz equivalent classes corresponding to $p \cdot I_1^+ + q \cdot I_1^-$.

Part I-(iii) Global monodromies up to stabilisation

Denis Auroux

Thm. A (Z.) Given a multi-set $\mathcal{O}$ of conjugacy classes of $SL(2,\mathbb{Z})$,
$\exists$ a tuple $(u_1,\dots,u_k)$ of positive Dehn twists such that,
for any torus fibrations $f_1:M_1\to S^2$ and $f_2:M_2\to S^2$ with $\mathcal{O}(f_1)=\mathcal{O}=\mathcal{O}(f_2)$,
for any global monodromies $(\phi_1,\dots,\phi_n)$ of f_1 and $(\psi_1,\dots,\psi_n)$ of f_2,
then

$$(\phi_1,\dots,\phi_n,u_1,\dots,u_k) \sim (\psi_1,\dots,\psi_n,u_1,\dots,u_k).$$

Remark: $(u_1,\dots,u_k)$ depends only on the non-simple part of $\mathcal{O}$.

Part I-(iv) The additional fibration tuple in Thm. A

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Def.: $[\phi]$ of $SL(2,\mathbb{Z})$ is **simple** if one of the following holds:

(0), $\mathrm{tr}(\phi)=0$

(1), $\mathrm{tr}(\phi)=\pm 1$

(2), $\mathrm{tr}(\phi)=\pm 2$ and ϕ is conjugate to one of $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 1 \\ & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & -1 \\ & -1 \end{bmatrix}$

(3), $\mathrm{tr}(\phi)=\pm 3$

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(3), $tr(\phi)=\pm 3$

Thm. B (Z.) When each $[\phi]\in\mathcal{O}$ is simple and not of trace ± 3,

$k=12$ and $(u_1,\dots,u_k) = (L,U,L,U,L,U,L,U,L,U,L,U) = (L,U)^6$.

Thm. C (Z.) When each $[\phi]\in\mathcal{O}$ is simple,

$k=60$ and $(u_1,\dots,u_k) = ((L,U)^6,(L,U,L)^4,(L,N,(L,U)^4,N,N)^3)$.

$$L=\begin{bmatrix}0&-1\\-1&2\end{bmatrix}\quad N=\begin{bmatrix}1&0\\1&1\end{bmatrix}\quad U=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$$

Part II-(i) Holomorphic fibrations

$2g-2+n>0, \quad h \geq 2$

Definition: A genus-h holomorphic **fibration** over a closed surface of genus g with n branch points is (Y,f,X) where

- Y : 2-dim closed complex manifold
- X : closed Riemann surface of genus g
- $f\colon Y \to X$: a holomorphic map branched over a set S of n points such that $f^{-1}(b)$ is a closed Riemann surface of genus h, for $b \in X \setminus S =: B$

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$$\mathrm{Fibr}(g,n,h) \underset{???}{\rightleftarrows} \mathrm{Fam}(g,n,h) \xrightarrow{MO} HM_{g,n,h} \subset M_{g,n,h}$$

$$(Y, f, X) \mapsto (C:=Y\setminus f^{-1}(S), f, B:=X\setminus S) \mapsto MO(f)$$

Part II-(ii) Classifying map & monodromy

$$\begin{array}{ccccc} \mathfrak{Fibr}(g,n,h) & \longrightarrow & \mathfrak{Fam}(g,n,h) & \xrightarrow{\mathrm{MO}} & HM_{g,n,h} \subset M_{g,n,h} \\ & & \Psi & & \\ & & (C,f,B) & \longmapsto & \mathrm{MO}(C,f,B) \end{array}$$

Isomorphic fibrations over X:
$(C_1,f_1,X)\sim(C_2,f_2,X)$ if $\exists$ a fibre-preserving biholomorphism $C_1 \cong C_2$

Isomorphic families over B:
$(C_1,f_1,B)\sim(C_2,f_2,B)$ if $\exists$ a fibre-preserving biholomorphism $C_1 \cong C_2$

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$$\mathcal{F}\mathrm{ibr}(g,n,h)/\sim \longrightarrow \mathcal{F}\mathrm{am}(g,n,h)/\sim \xrightarrow{\mathrm{MO}} HM_{g,n,h} \subset M_{g,n,h}$$

$$\begin{array}{ccc} \mathcal{F}\mathrm{am}(g,n,h) & & \\ \rotatebox{90}{$\in$} & & \\ (C,f,B) & \longmapsto & \mathrm{MO}(C,f,B) \end{array}$$

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$$\mathrm{Fibr}(g,n,h)/\sim \;\longrightarrow\; \mathrm{Fam}(g,n,h)/\sim \;\xrightarrow{MO}\; HM_{g,n,h} \subset M_{g,n,h}$$

$$\mathrm{Fam}(g,n,h)/\sim \;\ni\; (C,f,B) \longmapsto MO(C,f,B)$$

$$\mathrm{Fam}(g,n,h)/\sim \;\longrightarrow\; \{F : B \to \mathcal{M}_h\}$$

$$(C,f,B) \mapsto (B,F) \in \{F : B \to \mathcal{M}_h\}$$

Part II-(ii) Classifying map & monodromy

$$\begin{tikzcd}
\mathrm{Fibr}(g,n,h)/\sim \arrow[r] & \mathrm{Fam}(g,n,h)/\sim \arrow[r, "\mathrm{MO}"] \arrow[dd] & HM_{g,n,h} \subset M_{g,n,h} \\
& (C,f,B) \arrow[r, mapsto] & MO(C,f,B) \\
& \{F : B \to \mathcal{M}_h \text{ holomorphic}\} &
\end{tikzcd}$$

$(C,f,B) \in \mathrm{Fam}(g,n,h)$

$(C,f,B) \mapsto (B,F)$

$(B,F) \in \{F : B \to \mathcal{M}_h \text{ holomorphic}\}$

- classifying map F $: B \to M_h := T_h/\mathrm{Mod}_h$ is holomorphic

Part II-(ii) Classifying map & monodromy

$$\begin{tikzcd}
\mathrm{Fibr}(g,n,h)/\sim \arrow[r] & \mathrm{Fam}(g,n,h)/\sim \arrow[r, "MO"] \arrow[d, "CM"] & HM_{g,n,h} \subset M_{g,n,h} \\
 & G(g,n,h) &
\end{tikzcd}$$

$$(C,f,B) \in \mathrm{Fam}(g,n,h)/\sim \quad \longmapsto \quad MO(C,f,B)$$

$$(C,f,B) \mapsto (B,F) \in G(g,n,h)$$

$$G(g,n,h) := \{ F : B \to \mathcal{M}_h \text{ holomorphic} \}$$

- classifying map $F = CM(f) : B \to M_h := T_h / \mathrm{Mod}_h$ is holomorphic

- CM is finite-to-one

Part II-(ii) Classifying map & monodromy

$$\begin{tikzcd}
\mathrm{Fibr}(g,n,h)/\sim \arrow[r] & \mathrm{Fam}(g,n,h)/\sim \arrow[r, "MO"] \arrow[d, "CM"] & HM_{g,n,h} \subset M_{g,n,h} \\
& \mathcal{G}(g,n,h) \arrow[ur, "F \mapsto {[F_*]}"'] &
\end{tikzcd}$$

$$(C,f,B) \longmapsto MO(C,f,B)$$

$$(C,f,B) \mapsto (B,F)$$

$$\mathcal{G}(g,n,h) := \{F : B \to \mathcal{M}_h \text{ holomorphic}\}$$

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- $\tilde{F} : \mathbb{H}^2 \to \mathcal{T}_h \longrightarrow F_* := \pi_1(B) \to \pi_1^{orb}(\mathcal{M}_h) = \mathrm{Mod}_h$

Claim $F \mapsto [F_*]$ maps onto $HM_{g,n,h}$

Part II-(iii) Parshin-Arakelov finiteness

$2g-2+n>0, \quad h\geq 2$

- fibration/family (C,f,B) is isotrivial if $C_{b_1} \cong C_{b_2}$, $\forall$ generic $b_1, b_2 \in B$

Fix a closed Riemann surface X of genus g; fix the branch set S, $|S|=n$

P-A ver. 1: $\{$ non-isotrivial $(Y,f,X) \in \mathcal{F}ibr(g,n,h) \} /\sim$ is finite

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Uniform P-A: The above cardinality is bounded uniformly for $X \in \mathcal{M}_g$ and $|S|=n$ [Caporaso '02]

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P-A ver. 1: { non-isotrivial $(Y,f,X) \in \mathcal{F}ibr(g,n,h)$ } /~ is finite

Uniform P-A: The above cardinality is bounded uniformly for $X \in \mathcal{M}_g$ and |S|=n
[Caporaso '02]

Fix a Riemann surface B of type (g,n),

P-A ver. 2: { non-isotrivial $(C,f,B) \in \mathcal{F}am(g,n,h)$ } /~ is finite
[Imayoshi-Shiga '88]

Part II-(iii) Parshin-Arakelov finiteness

$2g-2+n>0, \quad h\geq 2$

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Rigidity Theorem: If $F_*=F'_*$ non-constant, then $F=F'$.

P-A ver.4: $HM_{g,n,h}|_B := \{ [F_*] \mid$ holomorphic $F:B\to\mathcal{M}_h \}$ is finite

Part II-(iv) Uniform bound for P-A Finiteness

Fix a Riemann surface B of type (g,n),

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Thm. G (Z.) Given $\varepsilon>0$, the following subset is finite.

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$$HM_{g,n,h}^{\geq\varepsilon} := \{ MO(f) \mid (C,f,B) \in \mathcal{F}am(g,n,h),\ sys(B)\geq\varepsilon \}$$

Def.: The image of a holomorphic map $F:B\to\mathcal{M}_h$
is called a **holomorphic curve** in $\mathcal{M}_h$.

Cor.: There are only finitely many holomorphic curves of type (g,n)
in $\mathcal{M}_h$ up to homotopy, when systole $\geq \varepsilon_0 > 0$.

Part II-(v) Proof of Thm. G: A glimpse of holomorphic curves

Thm. G (Z.) the following subset is finite

$$HM_{g,n,h}^{\geq\varepsilon} := \{\, MO(f) \mid (C,f,B) \in \mathcal{F}am(g,n,h),\ sys(B) \geq \varepsilon \,\}$$

Proof (Sketch):

Step 1: $\Sigma_{g,n} \to B$ sending each standard loop to a short loop.

Step 2: Irreducibility of $F_*(\pi_1(B))$
$\Longrightarrow sys(F(b)) \geq \varepsilon_0(g,n,h,\varepsilon)$ for some $b \in B_{cp}$.

Step 3: Finiteness. $\boxtimes$

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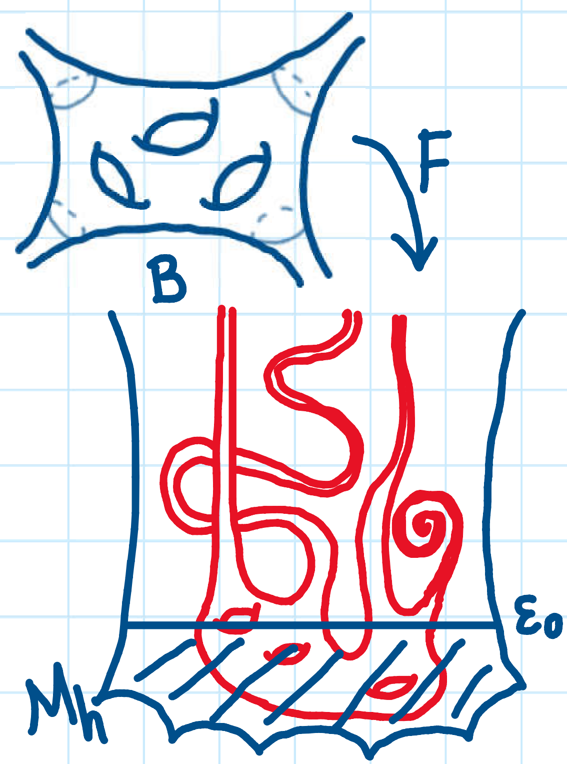

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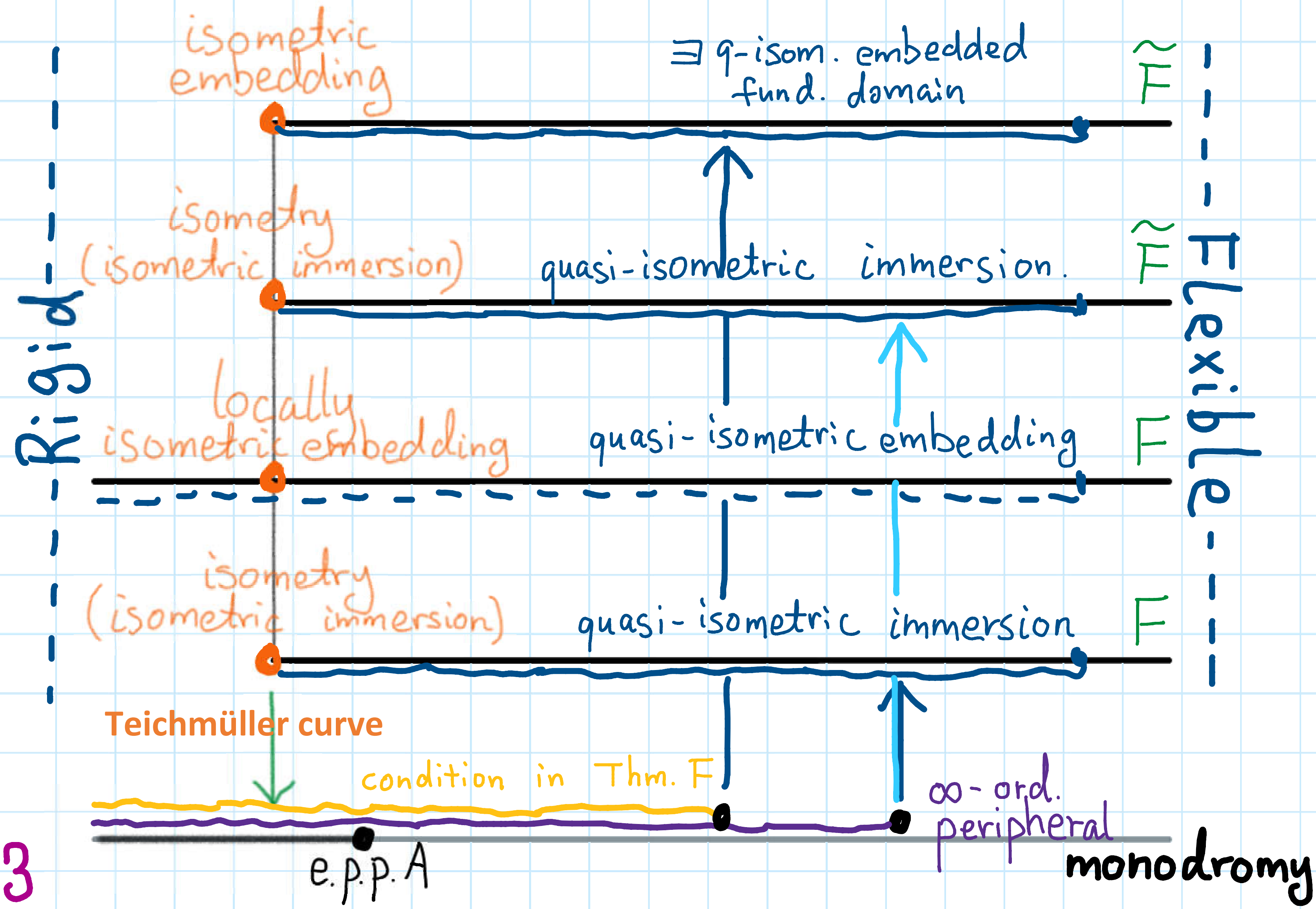
Part II-(vi) Shape of holomorphic curves

$F: B \longrightarrow \mathcal{M}_h$ holomorphic



Part II-(vi) Shape of holomorphic curves

The most rigid (holomorphic) curve :

Def.: A Teichmüller curve is the image of a holomorphic locally isometric map $F: (B, \frac{1}{2}dB) \to (\mathcal{M}_h, d_{\mathcal{M}})$

Remark: The lift is an isometric embedding

$$\tilde{F}: (H^2, \tfrac{1}{2}d_H) \to (\mathcal{T}_h, d_{\mathcal{T}})$$

- Such an isometric embedding is the SL(2,R)-orbit of a translation surface.
- The first Teichmüller curve was discovered by Veech [Veech '89]
- A Teichmüller curve is never complete (i.e. n>0)
- A Teichmüller curve is an algebraic curve defined over $\overline{\mathbb{Q}}$ [Möller '06]
- Every isometric map $\tilde{F}$ is holomorphic [Antonakoudis '15]

The monodromy of a Teichmüller curve is essentially purely pseudo-Anosov.

Part II-(vii) Cusp regions are quasi-isometrically embedded

- Teichmüller space: $\mathcal{T}_h$ Teichmüller distance: $d_{\mathcal{T}}$ $(\mathcal{T}_h, d_{\mathcal{T}})$

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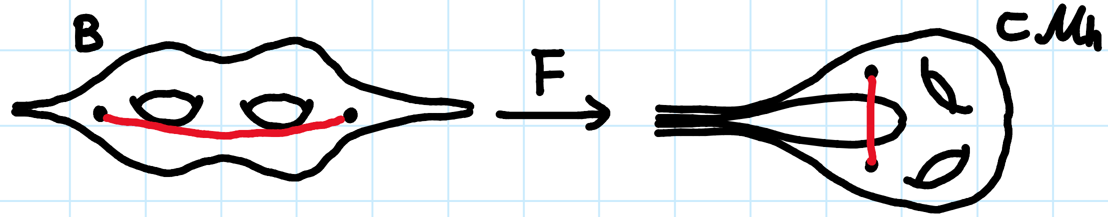
- F: $(B, \frac{1}{2}d_B) \to (\mathcal{M}_h, d_{\mathcal{M}})$



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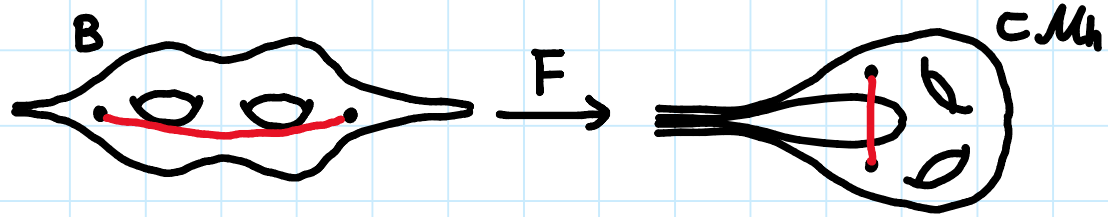

Thm. E-(1) (Z.) Let $F: B \to \mathcal{M}_h$ be a holomorphic map such that all peripheral monodromies are of ∞ order. Let $U \subset B$ be a cusp region. Then $F|_U$ is a $(1, K)$-quasi-isometric embedding, i.e.,

$$\frac{1}{2} d_B(b_1, b_2) \geq d_{\mathcal{M}}(F(b_1), F(b_2)) \geq \frac{1}{2} d_B(b_1, b_2) - K$$

for all $b_1, b_2 \in U$. Here $K = K(g, n, h, sys(B))$.

Part II-(viii) Holomorphic curves are quasi-isometrically immersed

$\frac{1}{2} d_B$ on B is induced by Kobayashi norm Kob_B

$d_{\mathcal{M}}$ on $\mathcal{M}_h$ is induced by Kobayashi norm $Kob_{\mathcal{M}}$

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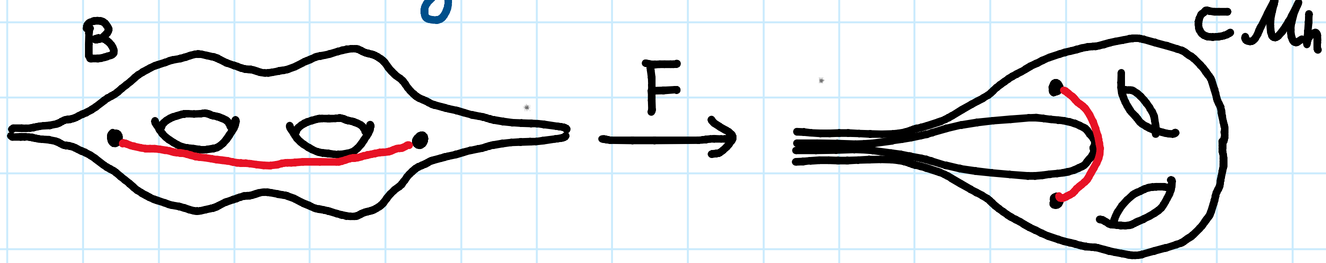
- path integral:

$$l_{\mathcal{M}}(F(\gamma)) := \int_0^1 \mathrm{Kob}_{\tau}\left(F\circ\gamma(t), \frac{d}{dt} F\circ\gamma(t)\right) dt \qquad \gamma : [0,1] \to B$$



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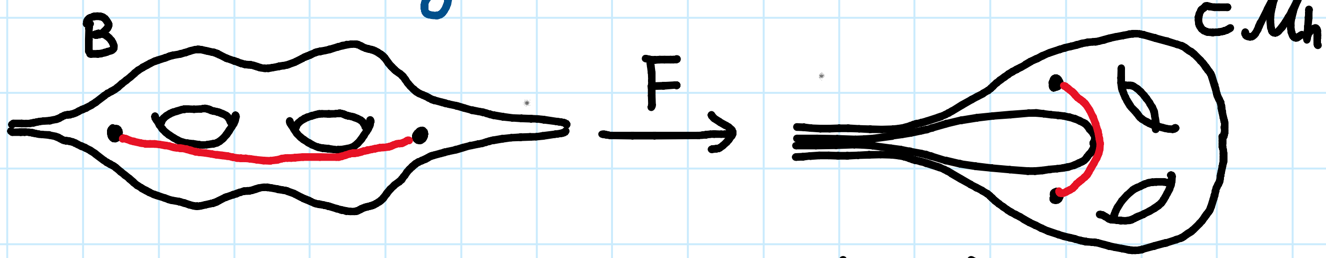

- (B, d_F) $\qquad d_F(b_1, b_2) := \inf\limits_{\gamma\subset B \text{ joining } b_1 \text{ to } b_2} \ell_{\mathcal{M}}(F(\gamma))$

Part II-(viii) Holomorphic curves are quasi-isometrically immersed

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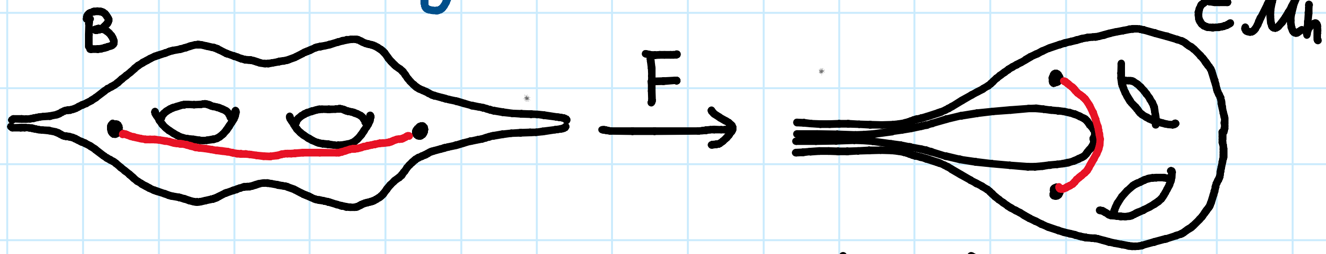


- (B, d_F) $\qquad d_F(b_1, b_2) := \inf\limits_{\gamma \subset B \text{ joining } b_1 \text{ to } b_2} \ell_{\mathcal{M}}(F(\gamma))$

Thm. E-(2) (Z.) Let $F : B \to \mathcal{M}_h$ be a holomorphic map such that all peripheral monodromies are of ∞ order.
Then F is a (1, K)-quasi-isometric immersion, i.e., for all $b_1, b_2 \in B$,

$$\frac{1}{2} d_B(b_1, b_2) \geq d_F(b_1, b_2) \geq \frac{1}{2} d_B(b_1, b_2) - K$$

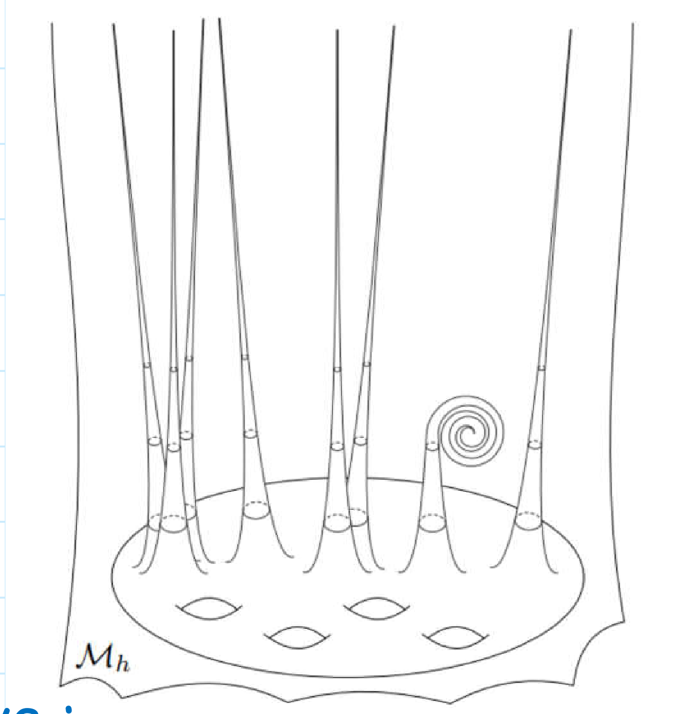
Part II-(ix) Holomorphic curves are quasi-isometrically immersed (cont'd)

Thm. E (Z.) If all peripheral monodromies are of ∞ order then

(1) $F|_{\text{a cusp region}}$ is a quasi-isometric embedding;

(2) F is a quasi-isometric immersion.

A better cartoon of the holomorphic curve :



F: B $\longrightarrow \mathcal{M}_h$ holomorphic

Part II-(ix) Holomorphic curves are quasi-isometrically immersed (cont'd)

Thm. E (Z.) If all peripheral monodromies are of ∞ order then

(1) $F|_{\text{a cusp region}}$ is a quasi-isometric embedding;

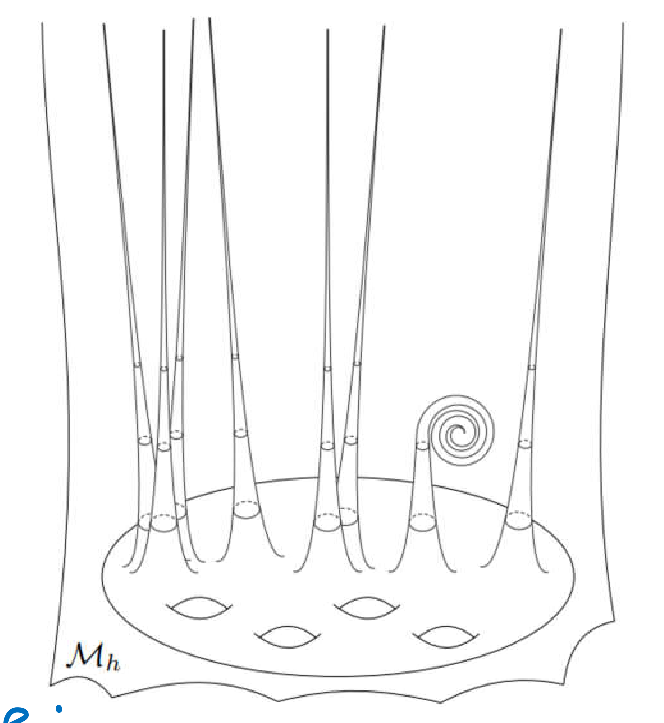
(2) F is a quasi-isometric immersion.

Remarks:

(1) When a peripheral monodromy is of finite order, the image of the cusp region might be contained in $\mathcal{M}_h^{\geqslant \varepsilon}$

(2) $\exists$ quasi-isometrically immersed but not isometrically immersed holomorphic curves in $\mathcal{M}_h$

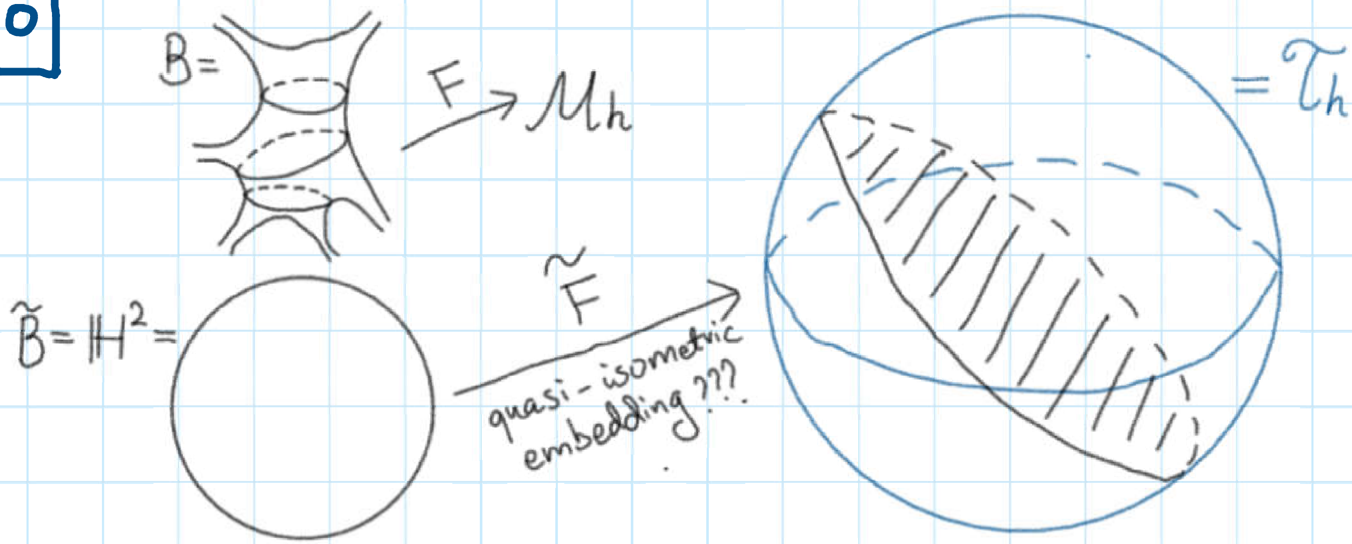
A better cartoon of the holomorphic curve :



$F: B \longrightarrow \mathcal{M}_h$ holomorphic

Part II-(x) Quasi-isometrically embedded fundamental domains

$g=0$



Part II-(x) Quasi-isometrically embedded fundamental domains

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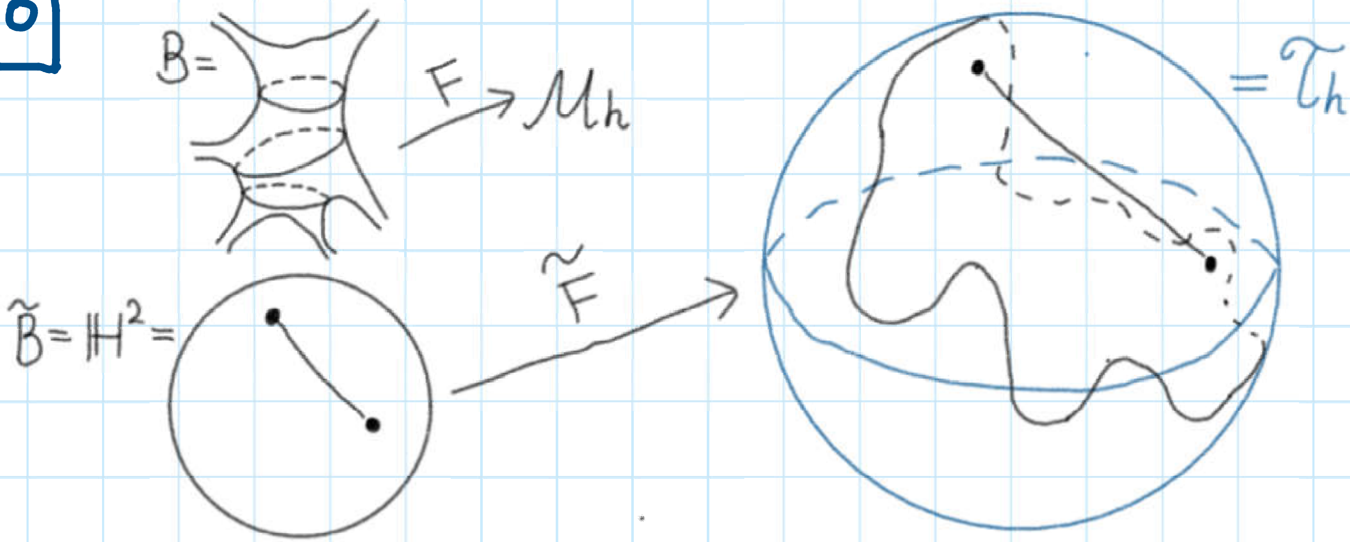

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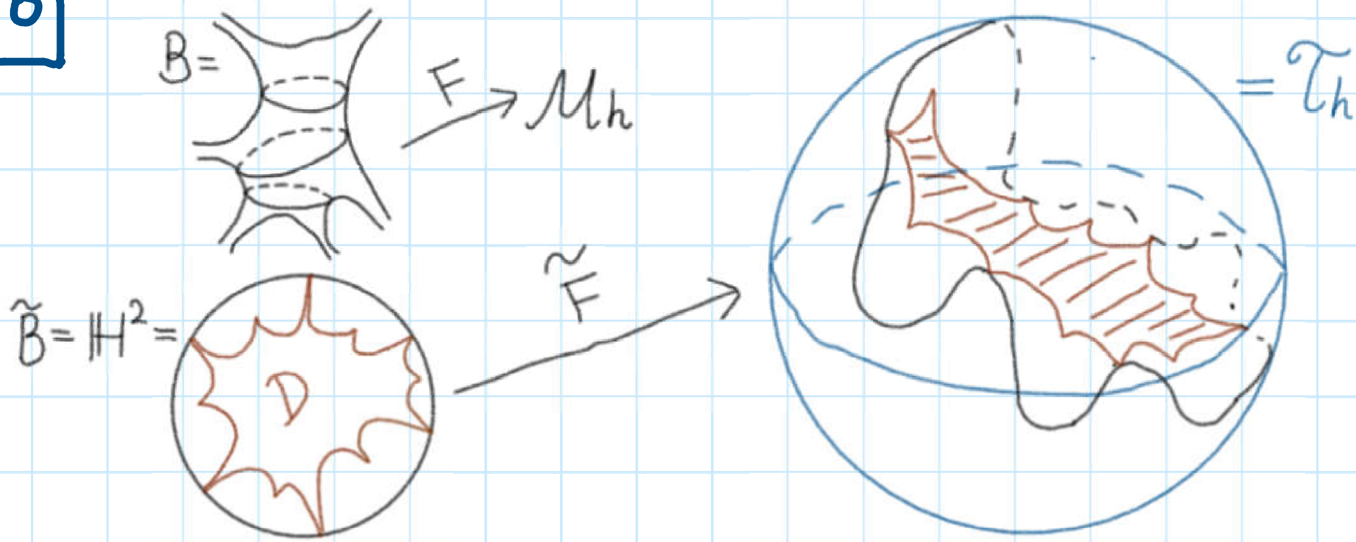

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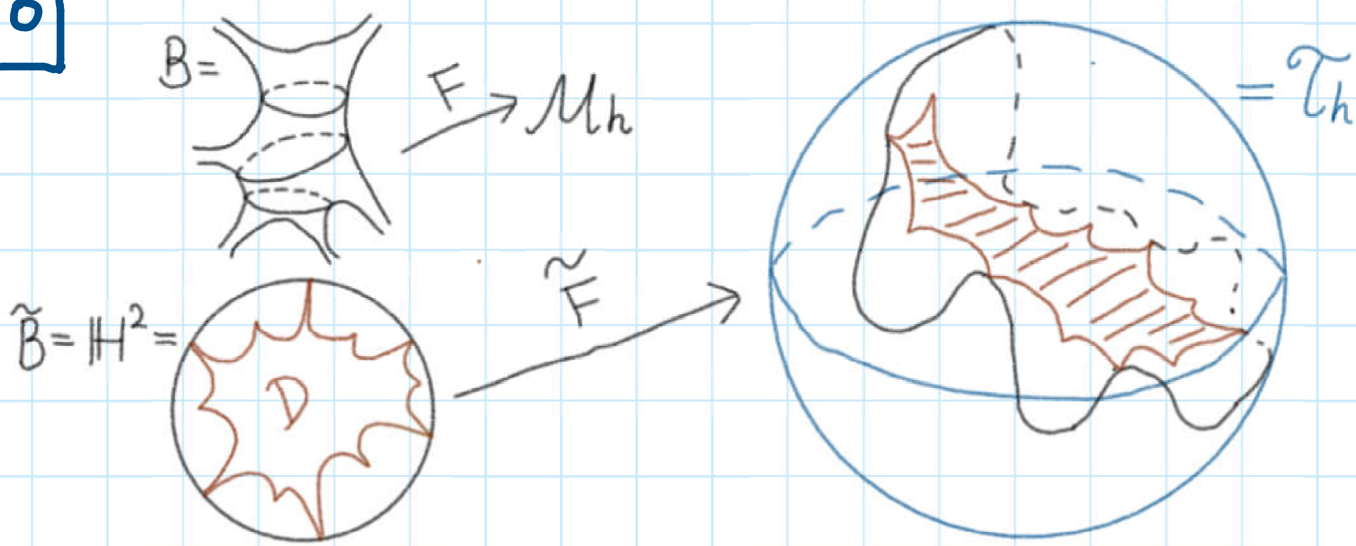


Cor. 3.4.8 (Z.)

Let $f: C^2 \to \mathbb{C}P^1 = X$ be a holomorphic genus-2 Lefschetz fibration without separating vanishing cycles.

Let $F: B \to \mathcal{M}_h$ be the classifying map of f.

Then, any lift of F has a quasi-isometrically embedded fundamental domain D of B.

Thank you for your attention!

Appendix 1 — A necessary condition for quasi-isom. embedded fundamental domains

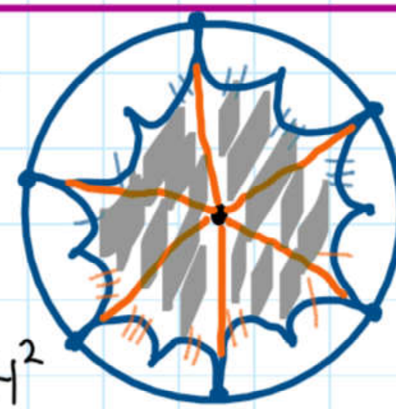
$g=0$ | $F: B \to \mathcal{M}_h$

$(B, \frac{1}{2} d_B) =$

monodromy: $(\phi_1, \dots, \phi_n)$

$\tilde{F}: \mathbb{H}^2 \to \mathcal{T}_h$

$(\tilde{B} = \mathbb{H}^2, \frac{1}{2} d_{\mathbb{H}})$



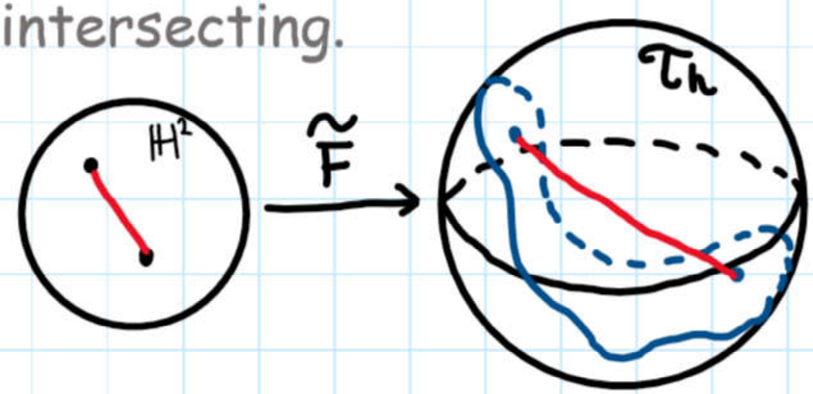
fundamental domain $D \subset \mathbb{H}^2$

Thm. F in the case g=0 (Z.) Let $F: B \to \mathcal{M}_h$ be a holomorphic map and $(\varphi_1, \dots, \varphi_n)$ be a global monodromy. Suppose that

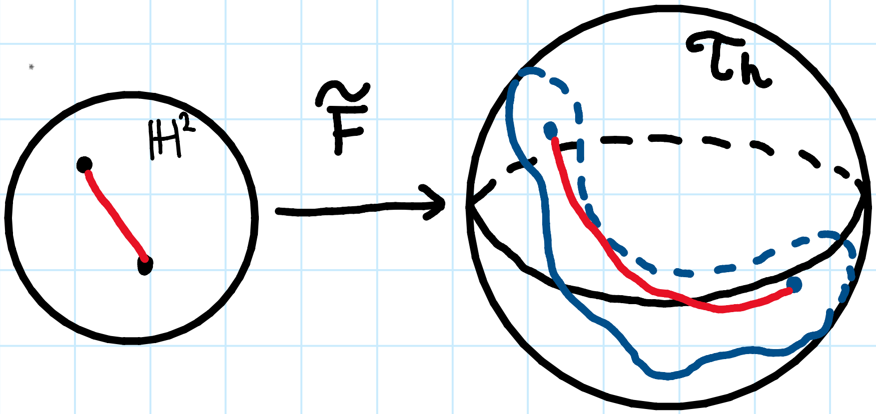
- Each φ_i is of ∞ order;
- $\phi_i^{\mu_i} = T_{\partial_i}$ is a multi-twist;
- Multi-curves $\partial_1, \dots, \partial_n$ are pairwise intersecting.

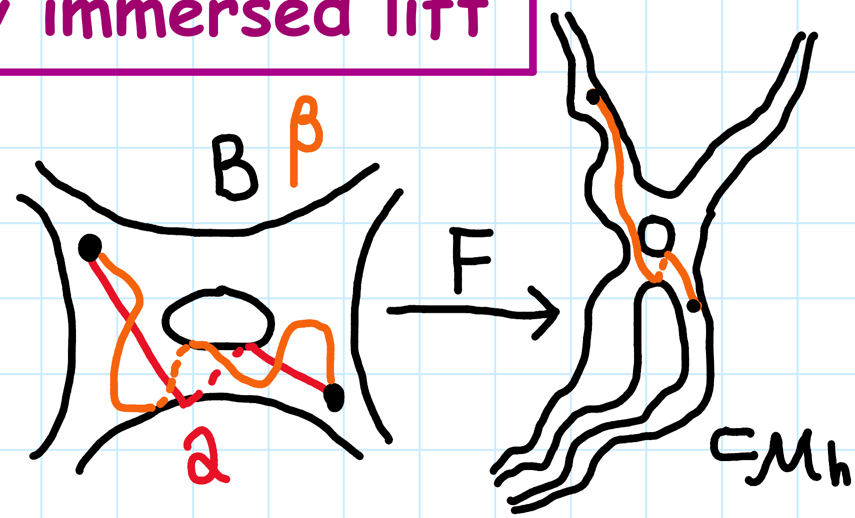
Then $\tilde{F}|_D : (D, \frac{1}{2} d_{\mathbb{H}}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ is a (2,K)-quasi-isometric embedding.

Here $K = K(g, n, h, sys(B))$.



Appendix 2 Quasi-isometrically immersed lift





Thm. Let $F: B \to \mathcal{M}_h$ be a holomorphic map s.t. all peripheral monodromies are of ∞ order. Then F is a (K,C)-quasi-isometric immersion, i.e., for any geodesic segment $\alpha \subset B$,

$$\ell_B(\alpha) \geq \ell_{\mathcal{T}}(F(\beta)) \geq \frac{1}{K}\ell_B(\alpha) - C$$

where β has the shortest image under F amount all paths relatively isotopic to α.

Appendix 3 Open questions

Classify elements of $M_{g,n,h}$:

- $M_{0,n,1}$: $\mathcal{O}(f)$ + extra invariants
- stabilisation : when $g \geqslant 2$, is there an additional tuple depending only on $\mathcal{O}(f)$?

Parshin – Arakelov finiteness:

- Is $HM_{g,n,h} := MO(\mathcal{F}am(g,n,h))$ finite ?
- Is $H_cM_{g,n,h} := MO(\mathcal{F}ibr(g,n,h))$ finite ?
- Given $m \in HM_{g,n,h}$, how to describe
$$\{B \in \mathcal{T}_{g,n} \mid \exists (C,f,B) \in \mathcal{F}am(g,n,h) \ MO(C,f,B) = m\} \subset \mathcal{T}_{g,n} \ ?$$

Shape of holomorphic curves in $\mathcal{M}_h$:

- When F_* is injective, is $\tilde{F}$ quasi-isom. embedded ?
- When F_* is e.p.p.A., is F a Teichmüller curve ?
- Given the shape of F in the sense of coarse geometry, how to describe the corresponding subset of $HM_{g,n,h}$?