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Variétés de dimension quatre admettant des fibrations

Manifolds of dimension four admitting fibrations

Présentée par :

Yibo ZHANG

Direction de thèse :

Louis FUNAR DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALPES	Directeur de thèse
Greg MCSHANE PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES	Co-directeur de thèse

Rapporteurs :

STEPAN OREVKOV
PROFESSEUR DES UNIVERSITES, UNIVERSITE DE TOULOUSE
JUAN SOUTO
DIRECTEUR DE RECHERCHE, CNRS BRETAGNE ET PAYS DE LA LOIRE

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ATHANASE PAPADOPOULOS, DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALSACE	Président
LOUIS FUNAR, DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALPES	Directeur de thèse
GREG MCSHANE, PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES	Co-directeur de thèse
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Variétés de Dimension Quatre Admettant Des Fibrations
Manifolds of Dimension Four Admitting Fibrations

Yibo Zhang
Institut Fourier, UMR 5582, Laboratoire de Mathématiques
Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France
email: yibo.zhang@univ-grenoble-alpes.fr

Résumé

Cette thèse présente des résultats de finitude et de rigidité pour les variétés de dimension 4 admettant des fibrations. Tout d'abord, nous étudions la *fibration du tore* sur la 2-sphère, à savoir une fibration dont la fibre générique est le tore. Le *type des singularités* est défini comme étant le multi-ensemble des classes de conjugaison des monodromies de fibres autour des fibres singulières. Nous montrons que, si deux fibrations de tore sur S^2 ont le même type des singularités, alors leurs monodromies globales sont Hurwitz-équivalentes après avoir effectué des sommes directes avec une certaine fibration de Lefschetz du tore. Cette fibration supplémentaire du tore de Lefschetz est universelle lorsque le type des singularités est "simple".

Deuxièmement, nous étudions la *fibration holomorphe*, à savoir une variété complexe de dimension 2 se projetant sur une surface de Riemann de manière holomorphe. L'application classifiante d'une fibration holomorphe après suppression des fibres singulières, lorsqu'elle n'est pas isotriviale, est une carte holomorphe non constante $F : B \to \mathcal{M}_h$. Ici B est une surface hyperbolique de type (g, n), $\mathcal{M}_h$ est l'espace de modules des surfaces de Riemann fermées de genre h et l'image $F(B)$ est appelée une *courbe holomorphe* dans $\mathcal{M}_h$. Nous montrons que, lorsque tous les monodromies périphériques sont d'ordre infini, la carte holomorphe est une immersion quasi-isométrique dont les paramètres ne dépendent que de g, n, h et de la systole de B. Lorsque les monodromies périphériques satisfont également une condition supplémentaire, nous trouvons un relèvement qui plonge quasi-isométriquement un polygone fondamental de la surface hyperbolique B dans l'espace de Teichmüller. De plus, nous améliorons le théorème de finitude de Parshin-Arakelov, en démontrant qu'il n'existe qu'un nombre fini d'homomorphismes de monodromie induits par des courbes holomorphes de type (g, n) dans $\mathcal{M}_h$ où la systole est bornée loin de 0, à une équivalence près.

Abstract

This thesis presents finiteness and rigidity results for 4-manifolds admitting fibrations. First, we study the *torus fibration* over the 2-sphere, namely, a fibration whose generic fibre is the torus. The *type of singularities* is defined to be the multi-set of conjugacy classes of fibre monodromies around singular fibres. We show that, if two torus fibrations over S^2 have the same type of singularities, then their global monodromies are Hurwitz equivalent after performing direct sums with a certain torus Lefschetz fibration. This additional torus Lefschetz fibration is universal when the type of singularities is "simple".

Second, we study the *holomorphic fibration*, namely, a 2-dimensional complex manifold projecting into a Riemann surface holomorphically. The classifying map of a holomorphic fibration after removing singular fibres, when it is non-isotrivial, is a non-constant holomorphic map $F : B \to \mathcal{M}_h$. Here B is a hyperbolic surface of type (g, n), $\mathcal{M}_h$ is the moduli space of closed Riemann surfaces of genus h and the image $F(B)$ is called a *holomorphic curve* in $\mathcal{M}_h$. We show that, when all peripheral monodromies are of infinite order, the holomorphic map is a quasi-isometric immersion with parameters depending only on g, n, h and the systole of B. When peripheral monodromies also satisfy an additional condition, we find a lift quasi-isometrically embedding a fundamental polygon of the hyperbolic surface B into the Teichmüller space. Besides, we improve the Parshin-Arakelov finiteness theorem, by proving that there are only finitely many monodromy homomorphisms induced by holomorphic curves of type (g, n) in $\mathcal{M}_h$ where systole is bounded away from 0, up to equivalence.

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Chapitre 1

Introduction

A 4-dimensional manifold admitting a fibration is fibred by surfaces, with a finite number of the fibres permitted to have singularities. Precisely, let M^4 be a closed, oriented 4-manifold and B be a closed, oriented surface of genus g. A *fibration* of M^4 over B is a continuous map $f : M^4 \to B$ for which there exists some finite set $S \coloneqq \{p_1, \ldots, p_n\} \subset B$, called the *branch set*, so that the restriction of f to $M' \coloneqq M^4 \setminus f^{-1}(S)$ is a locally trivial fibration over $B' \coloneqq B \setminus S$. This fibration is called a *genus-h fibration* when the generic fibre is a closed, oriented surface of genus h.

Let Σ_h be a closed, oriented smooth surface of genus $h \geq 2$. The *Teichmüller space* of Σ_h, denoted by $\mathcal{T}_h$, is the set of isotopy classes of complex structures on Σ_h. This space has a complex structure which is induced by an embedding $\mathcal{T}_h \hookrightarrow \mathbb{C}^{3h-3}$, due to Bers and Maskit (see [Ber70] and [Mas70]). The *Teichmüller distance* on $\mathcal{T}_h$ is the minimum dilatation of a quasiconformal diffeomorphism between two marked Riemann surfaces compatible with the markings, denoted by $d_{\mathcal{T}}$ in the sequel (see Subsection 3.1.1 for the precise definition).

The *mapping class group* of Σ_h, denoted by Mod_h, consists of all orientation-preserving diffeomorphisms of Σ_h up to isotopy. This group acts properly discontinuously on $\mathcal{T}_h$ and the quotient space, denoted by $\mathcal{M}_h$, is called the *moduli space*. Each mapping class in Mod_h is a holomorphic automorphism of $\mathcal{T}_h$ and an isometry for the Teichmüller distance.

Choose a base point $t \in B'$. The locally trivial fibration $f\mid_{M'}$ determines the *monodromy homomorphism* $\Phi_{f,t} : \pi_1(B', t) \to \mathrm{Mod}_h$ by identifying $f^{-1}(t)$ with Σ_h. Let $\Sigma_{g,n}$ be an n-punctured oriented surface of genus g and $s \in \Sigma_{g,n}$ be a fixed point. By identifying (B', t) with $(\Sigma_{g,n}, s)$ via a homeomorphism, we obtain a homomorphism in $\mathrm{Hom}(\pi_1(\Sigma_{g,n}, s), \mathrm{Mod}_h)$. Different choices of the base point $t \in B'$ and the homeomorphism $(B', t) \cong (\Sigma_{g,n}, s)$ do not change the corresponding class $\mathbf{MO}(M^4, f, B) = \mathbf{MO}(M', f, B')$ of the monodromy homomorphism in

$$M_{g,n,h} \coloneqq \mathrm{Mod}_{g,n} \setminus \mathrm{Hom}(\pi_1(\Sigma_{g,n}, s), \mathrm{Mod}_h) / \mathrm{Mod}_h$$

where $\mathrm{Mod}_{g,n} \coloneqq \mathrm{Mod}(\Sigma_{g,n})$ acts on the source by diffeomorphism and Mod_h acts on the target by conjugation. We write $M_n = M_{0,n,1}$.

Consider the case $g = 0$ then $f : M^4 \to B = \mathbb{S}^2$ is a genus-h fibration over the 2-sphere. We choose homotopy classes of loops $\gamma_1, \ldots, \gamma_n \subset \mathbb{S}^2$ based at t such that each loop γ_j goes around some branch point p_i exactly once clockwise and the fundamental group $\pi_1(\mathbb{S}^2 \setminus S, t)$ is generated by $\gamma_1, \ldots, \gamma_n$ with the relation $\gamma_1 \cdots \gamma_n = 1$. The monodromy $\phi_j \coloneqq \Phi_{f,t}(\gamma_j)$ is called the *fibre monodromy* around the singular fibre $f^{-1}(p_j)$, which satisfies $\phi_1 \cdots \phi_n = 1$. The n-tuple $(\phi_1, \ldots, \phi_n)$ in Mod_h is called the *global monodromy* of f. Thus $\mathbf{MO}(M^4, f, \mathbb{S}^2) \in M_{0,n,h}$ is uniquely determined by t, $\gamma_1, \ldots, \gamma_n$ and $(\phi_1, \ldots, \phi_n)$. However, dragging the base point t amounts to changing $(\phi_1, \ldots, \phi_n)$ by a diagonal (or simultaneous) conjugacy. Also a different choice of homotopy classes of $\gamma_1, \ldots, \gamma_n$ may change $(\phi_1, \ldots, \phi_n)$ by a sequence of elementary transformations (or Hurwitz moves ; see Subsection 2.1.1 for more details) :

$$(\ldots, \phi_i\phi_{i+1}\phi_i^{-1}, \phi_i, \ldots) \xleftarrow{L_i \coloneqq R_i^{-1}} (\ldots, \phi_i, \phi_{i+1}, \ldots) \xrightarrow{R_i} (\ldots, \phi_{i+1}, \phi_{i+1}^{-1}\phi_i\phi_{i+1}, \ldots), 1 \leq i \leq n-1$$

so that the resulting tuple is called *Hurwitz equivalent* to the global monodromy $(\phi_1, \ldots, \phi_n)$. The fundamental group $\pi_1(\mathbb{S}^2 \setminus S, t)$ is isomorphic to $\mathbb{F}_{n-1}$ and the orbit space $M_{0,n,h}$ also has the following group-theoretic interpretation. There is a left action of $\mathrm{Aut}(\mathbb{F}_n)$ on $\mathrm{Hom}(\mathbb{F}_n, \mathrm{Mod}_h)$ by precomposition with the inverse. Suppose that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a generating set of $\mathbb{F}_n$. Artin's representation embeds the braid group B_n on n strands as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. The subset $\mathrm{Hom}(\mathbb{F}_n/\langle\alpha_1 \cdots \alpha_n\rangle, \mathrm{Mod}_h)$ of $\mathrm{Hom}(\mathbb{F}_n, \mathrm{Mod}_h)$ is B_n-invariant and identified with $\mathrm{Hom}(\pi_1(\mathbb{S}^2 \setminus$

$S), \mathrm{Mod}_h)$, which inherits the action of B_n. We therefore have

$$M_{0,n,h} = B_n \backslash \mathrm{Hom}(\mathbb{F}_{n-1}, \mathrm{Mod}_h) / \mathrm{Mod}_h \,.$$

The B_n action induces an action of the sphere braid group $B_n(\mathbb{S}^2)$ on $\mathrm{Hom}(\mathbb{F}_{n-1}, \mathrm{Mod}_h)/\mathrm{Mod}_h$ coming from the natural mapping class group action on this set.

The element $\mathbf{MO}(M^4, f, B) \in M_{g,n,h}$ is an apparent invariant of the fibration in topology. Therefore the study of genus-h fibrations by means of their monodromy addresses two independent questions.

Question 1.0.1. How does an element in $M_{g,n,h}$ limit the corresponding genus-h fibration ?

Question 1.0.2. How do we characterise or classify elements in $M_{g,n,h}$?

A *Lefschetz fibration* is the simplest fibration, which is a smooth fibration and contains only one singularity in each singular fibre, each singularity admitting complex local coordinates (z_1, z_2) compatible with the orientation of M^4 such that the fibration is locally given by $f(z_1, z_2) = z_1^2 + z_2^2$. Lefschetz fibrations become very interesting because of Gompf's observation [Gom04] and Donaldson's result [Don99] which conclude that the study of Lefschetz fibrations is essentially equivalent to the study of symplectic manifolds. If one relaxes the orientation requirement for Lefschetz fibrations, the fibrations are *achiral Lefschetz fibrations.* It was noted by Gompf and Stipsicz that many 4-manifolds admit achiral Lefschetz fibrations, see [GS99, Section 8.4]. Etnyre and Fuller [EF06] showed that any smooth, closed, oriented 4-manifold, after surgery a framed circle, admits an achiral Lefschetz fibration over the 2-sphere.

In this thesis, we are mostly concerned with two different objects. The first object is the torus fibration, namely, a fibration whose generic fibre is the torus. The second object is the holomorphic fibration, namely, a 2-dimensional complex manifold projecting into a Riemann surface holomorphically.

1.1 Torus fibrations over the 2-sphere

Inspired by the classification of torus Lefschetz fibrations given by Moishezon and Livné [Moi77], Matsumoto's study on the global monodromies of torus achiral Lefschetz fibrations [Mat85] and Auroux's stabilisation for higher genus Lefschetz fibrations [Aur05], we first study torus fibrations over $\mathbb{S}^2$ and use the type of singularities to classify global monodromies, up to stabilisation.

The mapping class group $\mathrm{Mod}_1 = \mathrm{Mod}(\mathbb{T}^2)$ of torus is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. We will denote a multi-set by $[x_1, x_2, x_2, x_3, x_3, x_3, \ldots]$ and denote the conjugacy class of an element g in a group G by $Cl_G(g)$ (or $Cl(g)$ if we do not specify G).

Definition. Let $f : M^4 \to \mathbb{S}^2$ be a torus fibration over $\mathbb{S}^2$ with n branch points and $(\phi_1, \ldots, \phi_n)$ be a global monodromy of f. The *type* (of singularities) of f is defined to be the multi-set

$$\mathcal{O}(f) = [Cl_{\mathrm{SL}(2,\mathbb{Z})}(\phi_1), \ldots, Cl_{\mathrm{SL}(2,\mathbb{Z})}(\phi_n)],$$

which does not depend on the choice of its global monodromy.

For a torus Lefschetz fibration, the type of singularities depends only on the number of branch points, every ϕ_i being a positive Dehn twist around some simple loop. In this case, answers to both Question 1.0.1 and Question 1.0.2 are given by Moishezon and Livné : On the one hand, an orbit in M_n, if it does correspond to a torus Lefschetz fibration, determines the unique one up to fibre-preserving diffeomorphism (see Part II, Lemma 7a in [Moi77]). On the other hand, for torus Lefschetz fibrations with the same number of branch points, the action of B_n on the set of their monodromy homomorphisms is transitive (see Part II, Lemma 8 in [Moi77]). This result was generalized by Orevkov (see [Ore04]).

For a torus achiral Lefschetz fibration, we say that the orientation is still preserved for a type I_1^+ singular fibre but not for a type I_1^- singular fibre. In this case, the global monodromy was first investigated by Matsumoto in [Mat85] (see also [GS99, Section 8.4]). An inspirational result in his study introduces a representative of the global monodromy using elementary transformations which is, however, not unique. In particular, one cannot readily classify those torus achiral Lefschetz fibrations (or their corresponding elements in M_n) whose singular fibres of type I_1^+ and I_1^- occur in pairs.

In general, it is extremely difficult to classify the orbits in M_n. An algebraic understanding of M_n is related the study of Wiegold (see [Lub11]) who conjectured that

$$|\,\mathrm{Out}(\mathbb{F}_{n-1}) \backslash \mathrm{Epi}(\mathbb{F}_{n-1}, G) / \mathrm{Aut}(G)| = 1$$

for any finite simple group G and $n \geq 4$, where $\mathrm{Epi}(\mathbb{F}_{n-1}, G)$ denotes the set of epimorphisms $\mathbb{F}_{n-1} \to G$. For the study of its extension to surface groups, we refer to [FL18, Theorem 1.4].

As in [Aur05 ; CLP15 ; Sam20], we discuss the stable equivalence of algebraic objects by relating them to the direct sum construction. When the 2-sphere is replaced by an arbitrary surface, another notion of stabilisation corresponds to pinching a hole (see [CLP16 ; FP23]). For more interesting problems on the orbit space M_n and its variations, not related to the concept of stabilisation, we refer to [Aur06 ; Aur15].

Global monodromies with stabilisation

Suppose that $f_1 : M_1 \to \mathbb{S}^2$ and $f_2 : M_2 \to \mathbb{S}^2$ are two torus fibrations. Choosing a pair of 2-disks $D_1, D_2 \subset \mathbb{S}^2$ that do not contain any branch points of f_1, f_2 respectively, gluing $M_1 \setminus f_1^{-1}(D_1)$ and $M_2 \setminus f_2^{-1}(D_2)$ along some orientation reversing fibrewise homeomorphism $\beta : \partial f_1^{-1}(D_1) \to \partial f_2^{-1}(D_2)$, we obtain a fibre-connected sum $M_1 \oplus_\beta M_2$ between M_1 and M_2. The fibration f of $M_1 \oplus_\beta M_2$ piecing together f_1 and f_2 is again a torus fibration over $\mathbb{S}^2$, called a direct sum between f_1 and f_2 and written as $f = f_1 \oplus f_2$ if we do not specify β. In [Aur05] Auroux introduced the direct sum between a fibration and a fixed standard fibration, called stabilisation. He then proceeded to give a classification of genus $g \geq 2$ Lefschetz fibrations, up to stabilisation.

Definition. A conjugacy class of $\mathrm{SL}(2, \mathbb{Z})$ which either corresponds to elements of trace $0, \pm1, \pm3$ or else contains $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ is called *simple*.

The following result is a rather general extension of Auroux's stable classification in genus 1 but for arbitrary singularities :

Theorem A. *Let $\mathcal{O}$ be a multi-set of conjugacy classes of $\mathrm{SL}(2, \mathbb{Z})$. There exists a torus Lefschetz fibration $f^L_{\mathcal{O}}$ over $\mathbb{S}^2$ depending only on the non-simple conjugacy classes occurring in $\mathcal{O}$ that has the following property : for $i = 1, 2$,*
- *let f_i be a torus fibration over $\mathbb{S}^2$ with $\mathcal{O}(f_i) = \mathcal{O}$;*
- *let $\widetilde{f}_i$ be a direct sum between f_i and $f^L_{\mathcal{O}}$;*
- *let $(g_1^{(i)}, \ldots, g_n^{(i)})$ be a global monodromy of $\widetilde{f}_i$.*

Then $(g_1^{(1)}, \ldots, g_n^{(1)})$ and $(g_1^{(2)}, \ldots, g_n^{(2)})$ are Hurwitz equivalent i.e. one can transform $(g_1^{(1)}, \ldots, g_n^{(1)})$ into $(g_1^{(2)}, \ldots, g_n^{(2)})$ using a finite sequence of elementary transformations.

In Theorem A, the choices of direct sums $\widetilde{f}_1$, $\widetilde{f}_2$, base points and loops for the global monodromies are far from unique. As such, we adopt the following convention : we will use the double plural to highlight the unlimited objects, say all global monodromies of all direct sums.

Theorem A shows that, in particular, given a torus fibration f over $\mathbb{S}^2$, all global monodromies of all direct sums $f \oplus f^L_{\mathcal{O}(f)}$ are pairwise Hurwitz equivalent. The additional fibration $f^L_{\mathcal{O}}$ in Theorem A can be replaced by a torus fibration with fewer branch points but which is not a Lefschetz fibration (see Theorem 2.1.12 for a more detailed reformulation). In both cases, the number of branch points in the additional fibration depends on the number of non-simple elements in $\mathcal{O}$. In particular, we have the following results :

Theorem B. *There exists a torus Lefschetz fibration f^L_{12} over $\mathbb{S}^2$ with 12 branch points such that, for any multi-set $\mathcal{O}$ of simple conjugacy classes of $\mathrm{SL}(2, \mathbb{Z})$ corresponding to elements of trace $0, \pm1$ or ±2, all global monodromies of all direct sums $f \oplus f^L_{12}$ with f a torus fibration over $\mathbb{S}^2$ satisfying $\mathcal{O}(f) = \mathcal{O}$ are pairwise Hurwitz equivalent.*

Theorem C. *There exists a torus Lefschetz fibration f^L_{60} over $\mathbb{S}^2$ with 60 branch points such that, for any multi-set $\mathcal{O}$ of simple conjugacy classes of $\mathrm{SL}(2, \mathbb{Z})$, all global monodromies of all direct sums $f \oplus f^L_{60}$ with f a torus fibration over $\mathbb{S}^2$ satisfying $\mathcal{O}(f) = \mathcal{O}$ are pairwise Hurwitz equivalent.*

In Theorem B each of $-2, -1, 0, 1$ and 2 might occur as the trace of some element in $\mathcal{O}$. We emphasise that the "or" is always inclusive in this paper. The fibration f^L_{12} in Theorem B can be replaced by a non-Lefschetz fibration with only 6 branch points and the fibration f^L_{60} in Theorem C can be replaced by a fibration with only 19 branch points.

The stated Hurwitz equivalence in Theorem A, Theorem B and Theorem C is obtained with a specific normal form (see Theorem 2.1.12) which satisfies a remarkable property, called swappability (see Subsection 2.2.1). The normal form is computable : one can compute the finite sequence of elementary transformations with algorithms (see Section 2.5).

The Hurwitz equivalence fails without stabilisation or with an unreasonable stabilisation, see Subsection 2.4.1. The following theorem compares the (unstable) Hurwitz equivalence and the stable equivalence between global monodromies of torus achiral Lefschetz fibrations.

Theorem D. *For torus achiral Lefshetz fibrations with p singular fibres of type I_1^+ and q singular fibres of type I_1^-, we have the following statements.*
(i) After performing direct sums with f_{12}^L, all global monodromies are Hurwitz equivalent.
(ii) If $p \neq q$, then all global monodromies are Hurwitz equivalent.
(iii) If $p = q \geq 1$, then the global monodromies have infinitely many Hurwitz equivalent classes and there exists an explicit combinatorial classification.

As a consequence of Theorem B, we partially extend Kas' classification of elliptic surfaces up to diffeomorphism [Kas77] to a stable classification of their global monodromies. *Elliptic surfaces* over $\mathbb{CP}^1$ are proper holomorphic maps $f : S \to \mathbb{CP}^1$ between a complex surface S and $\mathbb{CP}^1$ such that the generic fibre is an elliptic curve. An elliptic surface is certainly a torus fibration whose singular fibres were classified by Kodaira in [Kod64 ; Kod66] ; the fibre monodromies are described in [Mir89].

Corollary A. *Let $f_1 : S_1 \to \mathbb{CP}^1$ and $f_2 : S_2 \to \mathbb{CP}^1$ be elliptic surfaces without multiple singular fibres, without singular fibres of type I_v or I_v^*, $v \geq 2$ in Kodaira's classification. Suppose that $\mathcal{O}(f_1) = \mathcal{O}(f_2)$. Then, all global monodromies of all direct sums $f_1 \oplus f_{12}^L$ and $f_2 \oplus f_{12}^L$ are pairwise Hurwitz equivalent.*

Fibre-preserving homeomorphisms

An element in M_n does not provide all the data about the fibration. In most cases, a torus fibration cannot be determined by its monodromy in any way. Additional restrictions and data on the local models at singularities are essential.

One remarkable encoding for the local model comes from King's classification in [Kin78 ; Kin97] of isolated singularities and the local study of singularities by Church and Timourian in [CT72 ; CT74], using this we study the so-called *singular fibrations*.

Roughly speaking by singular fibration we mean a smooth fibration with only finitely many singularities each having a "nice" neighbourhood (see Subsection 2.2.3 for a precise definition). Each singularity is then characterised by a local Milnor fibre which is a sub-surface of the generic fibre, a binding link K and an open book decomposition. Singular fibrations have been studied in [Loo71 ; Fun11 ; Fun22]. The local properties of their singularities are related to the corresponding fibred knots (see e.g. [BZ03]).

As an improvement of Proposition 2.1 in [Fun22] as well as a consequence of Theorem C and the swappability of the corresponding normal form, we have the following stable classification of singular fibrations based on the type of singularities up to fibre-preserving homeomorphism :

Corollary B. *Let $f_1 : M_1 \to \mathbb{S}^2$ and $f_2 : M_2 \to \mathbb{S}^2$ be torus singular fibrations with a single singularity in each singular fibre and with $\mathcal{O}(f_1) = \mathcal{O}(f_2)$. Suppose that each local Milnor fibre of singularities in f_1 and f_2 is either*
- *a surface of genus 0 with ≤ 2 boundary components, or*
- *a surface of genus 1 with only 1 boundary component.*

Let $\widetilde{f_1} = f_1 \oplus f_{60}^L : \widetilde{M_1} \to \mathbb{S}^2$ and $\widetilde{f_2} = f_2 \oplus f_{60}^L : \widetilde{M_2} \to \mathbb{S}^2$ be direct sums. Then $(\widetilde{M_1}, \widetilde{f_1})$ and $(\widetilde{M_2}, \widetilde{f_2})$ are fibre-preserving homeomorphic.

1.2 Holomorphic fibrations and their classifying maps

Siebert and Tian [ST05] proved that any genus-2 Lefschetz fibration without reducible fibres and satisfying a mild condition is holomorphic. They also conjectured in [ST99] that all hyperelliptic Lefschetz fibrations without reducible fibres are holomorphic. One may then require that a fibration is holomorphic, i.e., the total space $M^4 =: C$ is a 2-dimensional complex manifold, B is a Riemann surface of type (g, n) and $f : C \to B$ is a holomorphic map. By the complex version of the preimage theorem, each generic fibre $f^{-1}(b) =: C_b$ is a complex submanifold of C and hence a Riemann surface. As before, we suppose that the generic fibre is a closed surface of genus h. When the branch set $S = \emptyset$, the holomorphic fibration is further called a genus-h *holomorphic family.* Therefore, removing singular fibres of a holomorphic fibration, we obtain a holomorphic family of closed Riemann surfaces of genus h over a Riemann surface $B' := B \setminus S$ of type $(g, n + |S|)$. Fixing the marking on some fibre C_t with $t \in B'$, one may consider a marking on $f : C' := C \setminus f^{-1}(S) \to B'$

such that each fibre C_b is a marked Riemann surface of genus h and this marking is compatible with the localisation. We have thus associated with the marked family the unique canonical *classifying map* $\mathbf{CM}(C', f, B') : B' \to \mathcal{M}_h$. It is well-known [Gro62 ; Ear73] that these classifying maps are holomorphic and each could be the classifying map of at most

$$\#\operatorname{Hom}(\pi_1(B,t), \operatorname{Aut}(C_t)) \leq (2g+n)^{84(h-1)}$$

many non-isotrivial non-isomorphic holomorphic families, due to Hurwitz's $84(g-1)$ theorem. Here, a holomorphic family C/B is *isotrivial* if the fibres are all biholomorphic, i.e. $C_{b_1} \cong C_{b_2}$ for $b_1, b_2 \in B$. Two holomorphic families C/B and C'/B over B are *isomorphic* if there exists a biholomorphic map between C and C' that preserves fibrations.

Therefore the study of holomorphic families of Riemann surfaces is equivalent to the study of holomorphic curves in the moduli space of Riemann surfaces. We would always suppose that $2g-2+n > 0$ and $h \geq 2$ and consider the holomorphic map $F : B \to \mathcal{M}_h$ from a hyperbolic surface of type (g,n) to the moduli space of closed Riemann surfaces of genus h.

A *holomorphic disc* in $\mathcal{T}_h$ is the image of a holomorphic map $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ from the upper half plane $\mathbb{H}^2 \subset \mathbb{C}$ to the Teichmüller space. The hyperbolic plane is endowed with the usual complex structure making it biholomorphic to the open unit disc. Passing to the quotient we obtain a map $\mathring{F} : \mathbb{H}^2 \to \mathcal{M}_h$ to the moduli space. Let $\Gamma \leq \operatorname{Stab}(\mathring{F}) := \{\phi \in \operatorname{Aut}(\mathbb{H}^2) \mid \mathring{F} \circ \phi = \mathring{F}\}$ be a lattice and set $B := \Gamma \backslash \mathbb{H}^2$. Suppose that B is an oriented surface of genus g with n cusps (without boundary). The image of the quotient map $F : B \to \mathcal{M}_h$ is then a *holomorphic curve* of type (g,n) in $\mathcal{M}_h$.

We use d_B to denote the hyperbolic distance on B. The *systole* of B, denoted by $\operatorname{sys}(B)$, is the length of the shortest essential (i.e. non-contractible and non-peripheral) closed curve. A *cusp region* of B, usually denoted by U, is the neighbourhood of a cusp bounded by a horocycle of length 2.

As in [IS88, p.212] and [Moi77, p.176], a holomorphic map $F : B \to \mathcal{M}_h$ induces a group homomorphism $F_* : \pi_1(B,t) \to \operatorname{Mod}_h$ which is called a *monodromy homomorphism* of F (see Subsection 3.1.2 for the definition). We emphasise that there exists a holomorphic family $f : C \to B$ such that $F = \mathbf{CM}(C, f, B)$ and F_* is exactly a monodromy homomorphism of $f : C \to B$. We write $\mathbf{MO}(F) = \mathbf{MO}(C, f, B)$ for convenience. The image $F_*([\gamma])$ along a based loop $\gamma \subset B$ that goes once or several times around a cusp, clockwise or counterclockwise, is called a *peripheral monodromy* of the cusp.

As B and $\mathcal{T}_h$ are complex manifolds, both of them are endowed with intrinsic *Kobayashi pseudo-norms*, which need not be positive definite, as follows.

Definition 1.2.1. Let X be a complex manifold. The *Kobayashi pseudo-norm* $\operatorname{Kob}_X : TX \to \mathbb{R}_{\geq 0}$ on X is defined by $\operatorname{Kob}_X(x,v) = \inf_\phi\{1/c\}$ for $x \in X$ and $v \in T_xX$, where the infimum is taken over all holomorphic maps ϕ from the unit disc in $\mathbb{C}$ to X satisfying $\phi(0) = x$ and $(d\phi)_0(\partial/\partial z) = c \cdot v$.

A smooth manifold X with a pseudo-norm $TX \ni (x,v) \mapsto K(x,v)$ is also endowed with a pseudo-distance $d_{X,K} : X \times X \to \mathbb{R}$ given by the formula

$$d_{X,K}(x_1, x_2) = \inf_\gamma \int_0^1 K(\gamma(t), \dot{\gamma}(t))dt$$

where the infimum is taken over all piecewise smooth paths joining x_1 to x_2. The pseudo-distance $d_{X,\mathrm{Kob}}$ induced by Kob_X is called the *Kobayashi distance* on X. If Kob_X is a norm, i.e. it vanishes only at $0 \in TX$, then X is said to be *Kobayashi hyperbolic*.

Both $\mathbb{H}^2$ and $\mathcal{T}_h$ are Kobayashi hyperbolic. In fact, the Kobayashi (pseudo-)norm on $\mathbb{H}^2$ coincides with the norm of the Poincaré Riemannian metric or half the hyperbolic metric. The Kobayashi distance on $\mathbb{H}^2$ coincides with half the hyperbolic distance, i.e.

$$\operatorname{Kob}_{\mathbb{H}^2}(z,v) = \frac{1}{2}\frac{|dz(v)|}{\operatorname{Im}(z)} \quad \text{and} \quad d_{\mathbb{H}^2,\mathrm{Kob}}(z_1,z_2) = \tanh^{-1}\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|} = \frac{1}{2}d_{\mathbb{H}^2}(z_1,z_2)$$

for $z, z_1, z_2 \in \mathbb{H}^2$ and $v \in T_z\mathbb{H}^2$ (see [Aba89, Proposition 2.3.4]). On the other hand, the Kobayashi (pseudo-)norm $\operatorname{Kob}_{\mathcal{T}}$ on $\mathcal{T}_h$ is a Finsler metric and the Kobayashi distance on $\mathcal{T}_h$ coincides with the Teichmüller distance (see [Roy71, Theorem 3]), i.e. $d_{\mathcal{T},\mathrm{Kob}} = d_{\mathcal{T}}$.

Definition 1.2.2. Let $F : B \to \mathcal{M}_h$ be a holomorphic map whose lift is $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$.

- The distance $d_{\widetilde{F}}$ on $\mathbb{H}^2$ is induced by the pullback pseudo-norm $\widetilde{F}^* \operatorname{Kob}_{\mathcal{T}}$.
- The distance d_F on B is defined by

$$d_F(b_1, b_2) = \inf\{d_{\widetilde{F}}(\widetilde{b_1}, \widetilde{b_2}) \mid \widetilde{b_1} \in \mathbb{H}^2 \text{ is a lift of } b_1, \widetilde{b_2} \in \mathbb{H}^2 \text{ is a lift of } b_2\}.$$

- The *Teichmüller distance* $d_{\mathcal{M}}$ on $\mathcal{M}_h$ is defined by

$$d_{\mathcal{M}}(q_1, q_2) = \inf\{d_{\mathcal{T}}(\widetilde{q_1}, \widetilde{q_2}) \mid \widetilde{q_1} \in \mathcal{T}_h \text{ is a lift of } q_1, \widetilde{q_2} \in \mathcal{T}_h \text{ is a lift of } q_2\}.$$

To discuss the rigidity of $F : B \to \mathcal{M}_h$ and $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$, we recall the basic definition of a quasi-isometric embedding and define an extra notion of rigidity for $F : B \to \mathcal{M}_h$.

Definition 1.2.3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Given $\lambda \geq 1$ and $\epsilon \geq 0$, a map $f : X_1 \to X_2$ is called a (λ, ϵ)-*quasi-isometric embedding* if

$$d_1(x, y)/\lambda - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon$$

for all $x, y \in X_1$. In particular, when f is a $(1, 0)$-quasi-isometric embedding, we say that f is an *isometric embedding.*

Definition 1.2.4. Given $\lambda \geq 1$ and $\epsilon \geq 0$, a holomorphic map $F : B \to \mathcal{M}_h$ is called a (λ, ϵ)-*quasi-isometric immersion* if

$$(1/2)d_B(b_1, b_2)/\lambda - \epsilon \leq d_F(b_1, b_2) \leq (\lambda/2)d_B(b_1, b_2) + \epsilon$$

for all $b_1, b_2 \in B$. In this case, we say that $F(B)$ is *quasi-isometrically immersed.* In particular, when F is a $(1, 0)$-quasi-isometric immersion, we say that F is an *isometric immersion* and $F(B)$ is *isometrically immersed.*

Just as a quasi-isometric embedding needs not be an embedding, so a quasi-isometric immersion needs not be an immersion in the usual sense. However, an isometric immersion is indeed an immersion.

A holomorphic map $f : X_1 \to X_2$ between complex manifolds is distance-decreasing for the intrinsic Kobayashi distances, namely $d_{X_1,\mathrm{Kob}}(x, y) \geq d_{X_2,\mathrm{Kob}}(f(x), f(y))$ (see, e.g., [Aba89, Proposition 2.3.1]). In particular, the holomorphic map $F : B \to \mathcal{M}_h$ and its lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ satisfy the following inequalities :

$$\frac{1}{2}d_B\Big(b_1, b_2\Big) \geq d_F\Big(b_1, b_2\Big) \geq d_{\mathcal{M}}\Big(F(b_1), F(b_2)\Big),$$
$$\frac{1}{2}d_{\mathbb{H}^2}\Big(\widetilde{b_1}, \widetilde{b_2}\Big) \geq d_{\widetilde{F}}\Big(\widetilde{b_1}, \widetilde{b_2}\Big) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{b_1}), \widetilde{F}(\widetilde{b_2})\Big).$$

As a consequence, each peripheral monodromy must be either reducible or of finite order (see Corollary 3.1.4). In particular, if a peripheral monodromy ϕ is of infinite order, then a power ϕ^μ is a multi-twist.

An isometrically immersed curve $F(B) \subset \mathcal{M}_h$ is known as a *Teichmüller curve.* The first non-trivial cases were discovered by Veech ([Vee89]) and all Teichmüller curves in $\mathcal{M}_2$ and $\mathcal{M}_3$ are almost well-understood ([McM05 ; McM06] and [LN14 ; McM23, Theorem 5.5]). In $\mathcal{M}_h$ with $h \geq 5$, Teichmüller curves are elusive and the only known primitive case was given in [BM10].

Quasi-isometric rigidity for holomorphic curves

Every isometric immersion from B to $\mathcal{M}_h$ is either holomorphic or anti-holomorphic (see [Ant17]). However, a holomorphic map needs not be an isometric immersion. The following theorem shows that a weaker statement still holds.

Theorem E. *Given (g, n), h and ϵ with $2g - 2 + n > 0$, $h \geq 2$ and $\epsilon > 0$, there exists a constant $K = K(g, n, h, \epsilon)$ that depends only on g, n, h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) with $\mathrm{sys}(B) \geq \epsilon$ and cusp regions $U_1, \ldots, U_n \subset B$. Let $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$.*

(i) For each $i = 1, \ldots, n$, if a peripheral monodromy of the i-th cusp is of infinite order, then $F|_{U_i} : (U_i, (1/2)d_B) \to (\mathcal{M}_h, d_{\mathcal{M}})$ is a $(1, K)$-quasi-isometric embedding.

(ii) If all peripheral monodromies are of infinite order, then F is a $(1, K)$-quasi-isometric immersion.

Given a monodromy homomorphism F_*, if a peripheral monodromy of a fixed cusp is of infinite order, then all peripheral monodromies of this cusp are of infinite order. Therefore, the hypothesis in Theorem E - (ii) can be checked for only one peripheral monodromy of each cusp.

This result is optimal. On the one hand, there exists a holomorphic curve in $\mathcal{M}_h$ that is quasi-isometrically but not isometrically immersed, see Example 3.4.2. On the other hand, if peripheral monodromies of the i-th cusp are of finite order, then $F|_{U_i} : (U_i, (1/2)d_B) \to (\mathcal{M}_h, d_{\mathcal{M}})$ needs not be a quasi-isometric embedding. Example 3.4.5 shows such a possibility, when $F(U_i)$ is located in the thick part of $\mathcal{M}_h$.

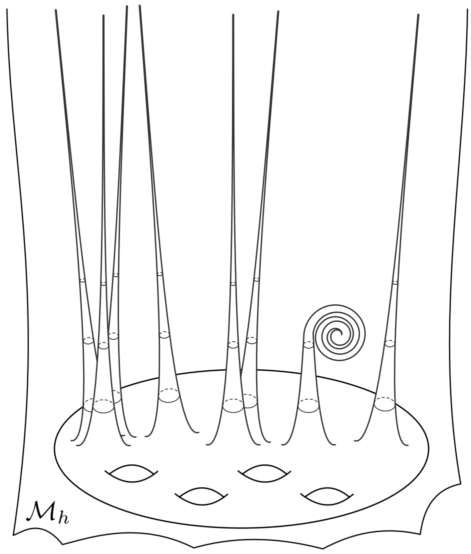


Figure 1.1 – A cartoon of a holomorphic curve in $\mathcal{M}_h$. Each cusp neighbourhood of the holomorphic curve almost keeps the hyperbolic structure along the Teichmüller metric, unless the peripheral monodromy is of finite order.

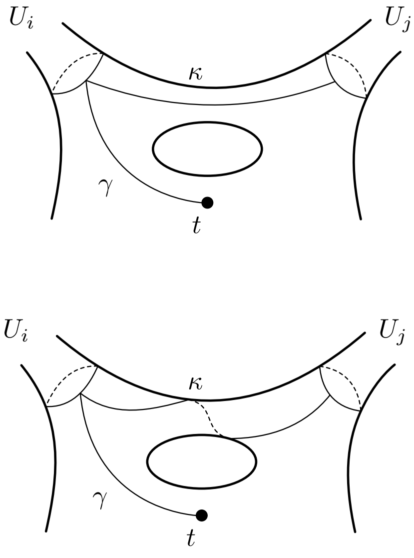


Figure 1.2 – Different geodesic segments joining the boundaries of two given cusp regions U_i and U_j, each providing a pair of peripheral monodromies (ϕ_i, ϕ_j).

Quasi-isometric rigidity for fundamental domains

If the holomorphic map $F : B \to \mathcal{M}_h$ is an isometric immersion, then the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ is a complex geodesic for the intrinsic Kobayashi norms. Teichmüller's uniqueness theorem (see, e.g., [FM11, Theorems 11.8 and 11.9]) shows that any two points of the Teichmüller space $\mathcal{T}_h$ are joined by a unique real geodesic. Therefore, the lift $\widetilde{F} : (\mathbb{H}^2, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ is an isometric embedding.

If the holomorphic map $F : B \to \mathcal{M}_h$ is a quasi-isometric immersion, in general, the lift $\widetilde{F} : (\mathbb{H}^2, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ fails to be a quasi-isometric embedding.

We now aim to obtain a hyperbolic polygon $D \subset \mathbb{H}^2$, i.e. a fundamental domain of B bounded by geodesic segments, such that $\widetilde{F}\Big|_D : (D, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_h, d_{\mathcal{T}})$ is a quasi-isometric embedding. Given a holomorphic map $F : B \to \mathcal{M}_h$ and a monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$, we start with a suitable condition on the monodromy.

Definition 1.2.5 (*disjointed* mapping classes)**.** Let ϕ_1 and $\phi_2 \in \mathrm{Mod}_h$ be reducible mapping classes. Suppose that there exist positive integers μ_1, μ_2 and multi-curves $\boldsymbol{\alpha_1} = \{\alpha_{1,1}, \ldots, \alpha_{1,m_1}\}$, $\boldsymbol{\alpha_2} = \{\alpha_{2,1}, \ldots, \alpha_{2,m_2}\}$ such that $\phi_1^{\mu_1}$, $\phi_2^{\mu_2}$ are multi-twists along $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$,

$$\phi_1^{\mu_1} = T_{\alpha_{1,1}}^{r_{1,1}} \circ \cdots \circ T_{\alpha_{1,m_1}}^{r_{1,m_1}}, \qquad \phi_2^{\mu_2} = T_{\alpha_{2,1}}^{r_{2,1}} \circ \cdots \circ T_{\alpha_{2,m_1}}^{r_{2,m_1}}$$

with $r_{i,j} \in \mathbb{Z} \setminus \{0\}$, for $i = 1, 2$ and $j = 1, \ldots, m_i$. We say that ϕ_1 and ϕ_2 are *disjointed* if pairs of curves in $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ are disjoint or coincide.

Definition 1.2.6 (*disjointed* peripheral monodromies)**.** Let U_i, U_j be cusp regions of the hyperbolic surface B, $i \neq j$, endowed with a geodesic segment κ joining ∂U_i to ∂U_j. Set $\{t_0\} = \partial U_i \cap \kappa$. and take an arbitrary path γ joining t to t_0 (see Figure 1.2). The loop along $\gamma \cup \partial U_i$ based at t that goes once around U_i clockwise is denoted by γ_i and its monodromy is denoted by ϕ_i. The loop along $\gamma \cup \kappa \cup \partial U_j$ based at t that goes once around U_j clockwise is denoted by γ_j and its monodromy is denoted by ϕ_j. We say that peripheral monodromies of U_i and U_j are *disjointed* along κ if ϕ_i and ϕ_j are reducible and disjointed.

A pair of disjointed mapping classes after a simultaneous conjugacy is again disjointed. Therefore, a pair of peripheral monodromies being disjointed along κ is independent of the choice of the path γ. However, a different choice of κ changes the peripheral monodromies by a non-simultaneous conjugacy.

Theorem F. *Given (g,n), h and ϵ with $2g-2+n>0$, $h\geq 2$ and $\epsilon>0$, there exists a constant $K=K(g,n,h,\epsilon)$ that depends only on g,n,h,ϵ and satisfies the following statement. Let $B=\Gamma\backslash\mathbb{H}^2$ be an oriented hyperbolic surface of type (g,n) with $\operatorname{sys}(B)\geq\epsilon$ and cusp regions $U_1,\ldots,U_n\subset B$. Let $D\subset\mathbb{H}^2$ be a fundamental convex polygon of B with exactly n ideal points. Let $F:B\to\mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_*\in\operatorname{Hom}(\pi_1(B,t),\operatorname{Mod}_h)$. If*

- *peripheral monodromies of all cusps are of infinite order,*
- *for each $i\neq j$, $i=1,\ldots,n$ and $j=1,\ldots,n$, there exists a geodesic segment $\kappa_{i,j}\subset B$ joining ∂U_i to ∂U_j with a lift $\widetilde{\kappa_{i,j}}\subset D$ such that peripheral monodromies of U_i and U_j are not disjointed along $\kappa_{i,j}$,*

then $\widetilde{F}\Big|_D:(D,(1/2)d_{\mathbb{H}^2})\to(\mathcal{T}_h,d_{\mathcal{T}})$ is a $(2,K+\operatorname{diam}(D))$-quasi-isometric embedding.

When $(g,n)=(0,n)$, recall that one can describe the monodromy homomorphism $F_*:\pi_1(B,t)\to\operatorname{Mod}_h$ by a global monodromy which is an n-tuple $(\phi_1,\ldots,\phi_n)$ in Mod_h such that $\phi_1\cdots\phi_n=1$. Therefore, there exists a fundamental polygon $D\subset\mathbb{H}^2$ of B and segments $\kappa_{i,j}\subset B$ together with lifts $\widetilde{\kappa_{i,j}}\subset D$ that satisfy the following equivalence : the peripheral monodromies of U_i and U_j are disjointed along $\kappa_{i,j}$ if and only if ϕ_i and ϕ_j are disjointed. Hence, the existence of a fundamental polygon D as in Theorem F such that $\widetilde{F}\Big|_D$ is a quasi-isometric embedding is equivalent to the existence of a tuple being Hurwitz equivalent to $(\phi_1,\ldots,\phi_n)$ whose components are of infinite order and pairwise non-disjointed.

The second hypothesis of Theorem F sometimes is a mild condition. In particular, a holomorphic curve of type $(0,n), n\geq 3$ in $\mathcal{M}_2$ with each peripheral monodromy the Dehn twist along a non-separating closed curve (i.e. the image of the classifying map of a genus-2 Lefschetz fibration without reducible fibres) must have a quasi-isometrically embedded fundamental polygon (see Subsection 3.4.3).

Finiteness result for holomorphic curves

The geometric Shafarevich conjecture, now known as the Parshin-Arakelov finiteness (see [Par68 ; Ara72 ; IS88]), claims that there are only finitely many non-isotrivial non-isomorphic holomorphic families of Riemann surfaces of genus h over a given base space B (assuming $h\geq 2$ and B hyperbolic).

The finiteness of holomorphic families corresponds to the finiteness of monodromy homomorphisms up to conjugacy. On the one hand, a class of homomorphisms in

$$M_h(B,t):=\operatorname{Hom}(\pi_1(B,t),\operatorname{Mod}_h)/\operatorname{Mod}_h$$

determines the topology of C/B. On the other hand, non-isotrivial holomorphic families over B having the same class of monodromy homomorphism in $M_h(B,t)$ are isomorphic (see Rigidity Theorem in [IS88]).

We aim to analyse holomorphic families over homeomorphic but non-biholomorphic Riemann surfaces, comparing their monodromy homomorphisms. Two holomorphic fibrations, even if non-holomorphic, correspond to the same class in $M_{g,n,h}$ if and only if they are isomorphic after removing singular fibres. The set $M_{g,n,h}$ is infinite. Moreover, at least for certain g, n and $h\geq 3$, there exist infinitely many symplectic Lefschetz fibrations with pairwise non-homeomorphic total spaces (see [FS04]). Therefore, the subset of classes realised by symplectic Lefschetz fibrations is also infinite. However in [Cap02] Caporaso proved that there is a uniform, i.e. independent of B, bound for the number of classes in $M_{g,n,h}$, that can be realised by a genus-h holomorphic fibration over a Riemann surface B of type $(g,0)$ having n branch points, see also [Hei04 ; Del16]. The following theorem is an algebraic improvement of the Parshin-Arakelov finiteness, which improves Corollary 2 in [Shi97].

Theorem G. *Given (g,n), h and ϵ with $2g-2+n>0$, $h\geq 2$ and $\epsilon>0$, then the subset*

$$\left\{ \boldsymbol{MO}(C,f,B) \middle| \begin{array}{l} B \textit{ is a Riemann surface of type } (g,n) \textit{ with } \mathrm{sys}(B)\geq\epsilon \\ f: C\to B \textit{ is a non-isotrivial genus-h holomorphic family over } B \end{array} \right\} \subset M_{g,n,h}$$

is finite.

Remark 1.2.7. The $\mathbf{MO}(C,f,B)$ in Theorem G can be replaced with either $\mathbf{MO}(\mathbf{CM}(C,f,B))$ or $\mathbf{MO}(F)$ where $F: B\to\mathcal{M}_h$ is a non-constant holomorphic map, due to the many-to-one correspondence between holomorphic families and their classifying maps.

The following finiteness result is an immediate consequence of the above theorem.

Corollary 1.2.8 (Theorem 6.5 in [Shi14])**.** *Given (g,n) and h with $2g-2+n>0$ and $h\geq 2$, there are only finitely many Teichmüller curves of type (g,n) in $\mathcal{M}_h$.*

Two holomorphic curves $F_1: B_1\to\mathcal{M}_h$ and $F_2: B_2\to\mathcal{M}_h$ are called *homotopic* if B_1, B_2 are of the same type (g,n) and there exist orientation preserving diffeomorphisms $f_1:\Sigma_{g,n}\to B_1$, $f_2:\Sigma_{g,n}\to B_2$ such that $F_1\circ f_1$, $F_2\circ f_2$ are homotopic. Theorem G implies that there are only finitely many holomorphic curves of type (g,n) in $\mathcal{M}_h$ up to homotopy, when hyperbolic systoles are bounded away from 0.

1.3 Outline of the thesis

The thesis is roughly a combination of the author's papers [Zha23] and [Zha24].

We investigate torus fibrations in Chapter 2. Section 2.1 presents the main technique used for the study of torus fibrations over the 2-sphere and more detailed statements of our results for global monodromies of torus fibrations up to fibre sum stabilisation. In particular, Subsection 2.1.1 introduces the elementary transformation of global monodromies and some preliminaries. Subsection 2.1.2 provides a new technique in the study of global monodromies. The additional torus Lefschetz fibration used in the stabilisation is precisely given in Subsection 2.1.3. Subsection 2.1.4 claims that every stabled global monodromy can be transformed into a certain normal form, which implies Theorem A.

Section 2.2 introduces the so-called swappability of the normal form from a stable global monodromy. This property allows us to investigate fibre-preserving homeomorphisms between torus fibrations in Subsection 2.2.2. Subsection 2.2.3 introduces singular fibrations using a remarkable encoding for the local model of singularities introduced by King, and illustrates Corollary B.

Moishezon and Livné [Moi77] introduced some inspirational results for tuples in $\mathrm{PSL}(2,\mathbb{Z})$. Our study of global monodromies strongly relies on two different extensions of their studies which we leave in Section 2.3.

Section 2.4 shows that the stabilisation used for global monodromies of torus fibrations has a great effect on the Hurwitz equivalence. In particular, Subsection 2.4.2 compares the stable classification and the unstable classification of achiral Lefschetz fibrations. All Hurwitz equivalences occurring in Theorem A, Theorem B and Theorem C are computable and the computability is discussed in Section 2.5.

We investigate holomorphic fibrations in Chapter 3. Subsections 3.1.1 and 3.1.2 introduces the main notions needed for studying the Teichmüller space and monodromies of a holomorphic map $F: B\to\mathcal{M}_h$. Subsection 3.1.3 presents some condition on the monodromy, which partially describes a Teichmüller curve and is used to prove Corollary 1.2.8. Subsection 3.1.4 revisits Mumford's compactness of thick moduli spaces and provides some tools that we will need later.

Subsection 3.1.5 revisits the complex structure on $\mathcal{T}_h$ and provides the *irreducibility* of $F: B\to\mathcal{M}_h$. Using this irreducibility, we provide an auxiliary result, i.e. Theorem 3.2.1, which claims that each non-constant holomorphic curve has a non-empty intersection with a certain thick part of the moduli space. The proof of Theorem G relies on this result and can also be found in Section 3.2.

For our rigidity results, we first investigate a holomorphic hyperbolic cusp region in $\mathcal{M}_h$ in Subsection 3.3.1. We emphasise that a mapping class of infinite order changes a marked hyperbolic surface slightly in $(\mathcal{T}_h, d_{\mathcal{T}})$ only if the mapping class is a multi-twist along small closed geodesics on the hyperbolic surface. This fact implies that a holomorphic hyperbolic cusp region in $\mathcal{M}_h$ is affected by a "force" from the cusp, which partially shows the quasi-isometric embedding in Theorem E - (i). Theorem 3.2.1 pulls the holomorphic curve as well as each holomorphic hyperbolic cusp region using another "force" from the thick part of the moduli space, hence we prove Theorem E in Subsection 3.3.2. The proof of Theorem F appears in Subsection 3.3.3. Finally, Section 3.4 provides some examples and an application on genus-2 Lefschetz fibrations.

Chapitre 2

Stable classification of torus fibrations over the 2-sphere

2.1 Connected sums and Hurwitz equivalence

2.1.1 Elementary transformations

We first define the elementary transformations. Throughout this subsection, G is an arbitrary group and $Z(G)$ is the center of G. An n-tuple in G is a sequence $(g_1, \ldots, g_n)$ of elements in G, each g_i is called a component of the tuple. Let $\mathcal{T}_{G,n}$ be the set of n-tuples $(g_1, \ldots, g_n)$ in G satisfying $g_1 \cdots g_n \in Z(G)$.

Definition 2.1.1. For $1 \leq i \leq n-1$, the *elementary transformations* (or *Hurwitz moves*) R_i is a bijection on the set of n-tuples in G defined by :

$$R_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n).$$

Both R_i and its inverse R_i^{-1} are elementary transformations. A pair of tuples $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ which can be transformed into each other by a finite sequence of elementary transformations are called *Hurwitz equivalent*, written as :

$$(g_1, \ldots, g_n) \sim (h_1, \ldots, h_n).$$

We emphasise that the set of all n-tuples in G can also be interpreted as $\mathrm{Hom}(\mathbb{F}_n, G)$ and the subset $\mathcal{T}_{G,n}$ is invariant under the elementary transformations.

Lemma 2.1.2. *For $(g_1, \ldots, g_n) \in \mathcal{T}_{G,n}$ and any $1 \leq k \leq n$, the tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to $(g_k, g_{k+1}, \ldots, g_n, g_1, g_2, \ldots, g_{k-1})$.*

Proof. Applying $R_{n-1} \circ \ldots \circ R_1$ on the n-tuple $(g_1, \ldots, g_n)$ we get $(g_2, \ldots, g_n, g_1)$. □

Let $\bullet$ denote the concatenation of tuples : $(g_1, \ldots, g_n) \bullet (h_1, \ldots, h_m) = (g_1, \ldots, g_n, h_1, \ldots, h_m)$. The power of a tuple corresponds to a repeated concatenation with itself. The symbol $\prod$ represents the concatenation of a family of tuples.

Lemma 2.1.3. *Let $(g_1, \ldots, g_n, h_1, \ldots, h_m, g_{n+1}, \ldots, g_{n+n'})$ be an $(n+m+n')$-tuple in G satisfying $h_i \cdots h_m \in Z(G)$. For $0 \leq k \leq n+n'$, this $(n+m+n')$-tuple is Hurwitz equivalent to*

$$(g_1, \ldots, g_k, h_1, \ldots, h_m, g_{k+1}, \ldots, g_{n+n'}). \tag{2.1}$$

In particular, let $(g_{1,1}, \ldots, g_{1,n_1})$, $(g_{2,1}, \ldots, g_{2,n_2})$, $\ldots$, $(g_{k,1}, \ldots, g_{k,n_k})$ be tuples in G satisfying $g_{j,1} \cdots g_{j,n_j} \in Z(G)$ for each of $j = 1, \ldots, k$. Then their concatenations in any order are pairwise Hurwitz equivalent.

Proof. Applying $R_{n+m-1} \circ \ldots \circ R_n$ if $n' > 0$ and applying $R_{n+1}^{-1} \circ \ldots \circ R_{n+m}^{-1}$ if $n > 0$ on the $(n+m+n')$-tuple we transform the tuple into $(g_1, \ldots, g_{n+1}, h_1, \ldots, h_m, g_{n+2}, \ldots, g_{n+n'})$ and $(g_1, \ldots, g_{n-1}, h_1, \ldots, h_m, g_n, \ldots, g_{n+n'})$ respectively. □

Lemma 2.1.4. *Let $(g_1, \ldots, g_n)$ be an n-tuple in G satisfying $g_i = g_j h$ with some $1 \leq i < j \leq n$ and h in $Z(G)$. Then $(g_1, \ldots, g_n) \sim (g_1, \ldots, g_{i-1}, g_j, g_{i+1}, \ldots, g_{j-1}, g_i, g_{j+1}, \ldots, g_n)$.*

Proof. Applying $R_{j-1}^{-1} \circ \ldots \circ R_{i+1}^{-1} \circ R_i \ldots \circ R_{j-1}$ on $(g_1, \ldots, g_n)$ we get the tuple

$$(g_1, \ldots, g_{i-1}, g_j, g_i g_j^{-1} g_{i+1} g_j g_i^{-1}, \ldots, g_i g_j^{-1} g_{j-1} g_j g_i^{-1}, g_i, g_{j+1}, \ldots, g_n),$$

which is equal to $(g_1, \ldots, g_{i-1}, g_j, g_{i+1}, \ldots, g_{j-1}, g_i, g_{j+1}, \ldots, g_n)$, as desired. □

Definition 2.1.5. An n-tuple in G is said to *contain a generating set* if its components form a generating set of the group G.

For instance, the modular group $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{+I, -I\}$ has the presentation

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle a, b \mid a^3 = b^2 = 1 \rangle;$$

both $(a^2b, ba^2, a^2b, ba^2, a^2b, ba^2)$ and (ba, ab, ba, ab, ba, ab) contain generating sets.

Lemma 2.1.6. *Suppose that $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_m)$ are tuples in G such that $(h_1, \ldots, h_m)$ contains a generating set. Let Q be an arbitrary element in G. If there exists a sub-tuple of $(g_1, \ldots, g_n)$, say $(g_l, \ldots, g_r)$ with $1 \leq l \leq r \leq n$, such that $\prod_{i=l}^{r} g_i \in Z(G)$, then the concatenation $(g_1, \ldots, g_n) \bullet (h_1, \ldots, h_m)$ is Hurwitz equivalent to*

$$(g_1, \ldots, g_{l-1}, Q^{-1} g_l Q, \ldots, Q^{-1} g_r Q, g_{r+1}, \ldots, g_n) \bullet (h_1, \ldots, h_m).$$

Proof. We express a given element Q in G as $q_1 \cdots q_u$ such that $q_i \in \{h_1, h_1^{-1}, \ldots, h_m, h_m^{-1}\}$, $i = 1, \ldots, u$. The lemma follows from Lemma 2.1.3 and the following substitutions via elementary transformations for each of $j = 1, \ldots, m$:

$$\begin{aligned}
&(g_1, \ldots, g_l, \ldots, g_r, \ldots, g_n) \bullet (h_1, \ldots, h_j, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, g_{r+1}, \ldots, g_n, h_1, \ldots, h_{j-1}, g_l, \ldots, g_r, h_j, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, g_{r+1}, \ldots, g_n, h_1, \ldots, h_j, h_j^{-1} g_l h_j, \ldots, h_j^{-1} g_r h_j, h_{j+1}, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, h_j^{-1} g_l h_j, \ldots, h_j^{-1} g_r h_j, g_{r+1}, \ldots, g_n) \bullet (h_1, \ldots, h_j, \ldots, h_m); \\
&(g_1, \ldots, g_l, \ldots, g_r, \ldots, g_n) \bullet (h_1, \ldots, h_j, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, g_{r+1}, \ldots, g_n, h_1, \ldots, h_j, g_l, \ldots, g_r, h_{j+1}, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, g_{r+1}, \ldots, g_n, h_1, \ldots, h_{j-1}, h_j g_l h_j^{-1}, \ldots, h_j g_r h_j^{-1}, h_j, \ldots, h_m) \\
&\to (g_1, \ldots, g_{l-1}, h_j g_l h_j^{-1}, \ldots, h_j g_r h_j^{-1}, g_{r+1}, \ldots, g_n) \bullet (h_1, \ldots, h_j, \ldots, h_m).
\end{aligned}$$

□

2.1.2 Contraction and restoration on tuple

In this subsection, we introduce the notions of *contraction* and *restoration* on tuples. We move on to a procedure that involves a series of operations, including contractions, restorations, and elementary transformations. The procedure behaves like a self-consistent machine, maintaining data about the given tuple and operations. Our study repeatedly utilises this procedure. To make it clear and easy to visualise, thus we start with the following definition.

Definition 2.1.7. An *iterated tuple* of height 0 in G is an element $g \in G$; for $h \geq 1$, an iterated tuple of height h in G is a tuple whose components are iterated tuples of height smaller than h such that at least one component is of height $h - 1$.

Take $g \in G$ and $H = (H_1, \ldots, H_n)$ an iterated tuple of height $h \geq 1$. The *evaluation* on an iterated tuple is defined by $ev(g) = g$ and $ev(H) = \prod_{i=1}^{n} ev(H_i)$. With $v \in G$, using the notation $g^v = v^{-1} g v$ we define H^v as

$$H^v = (H_1, \ldots, H_n)^v = (H_1^v, \ldots, H_n^v).$$

The elementary transformation R_i acts on the set of iterated tuples with $n \geq i + 1$ components by taking the conjugation of each element in H_i with $ev(H_{i+1})$ and swapping the positions, to wit

$$R_i(H_1, \ldots, H_n) = (H_1, \ldots, H_{i-1}, H_{i+1}, H_i^{ev(H_{i+1})}, H_{i+2}, \ldots, H_n).$$

Given an n-tuple $(g_1, \ldots, g_n)$ in G, we keep hold of the following data :

- $(h_1, \ldots, h_m)$: an tuple in G ;
- $(H_1, \ldots, H_m)$: an iterated tuple in G such that $(ev(H_1), \ldots, ev(H_m)) = (h_1, \ldots, h_m)$;

- $\mathcal{F}$: an ordered list such that each element is either
  - a pair (μ, σ) with $\mu \in \mathbb{Z}$ and σ an elementary transformation on (iterated) μ-tuples, or
  - a pair of integers (l, r) with $1 \leq l < r$.

At the beginning, both $(h_1, \ldots, h_m)$ and $(H_1, \ldots, H_m)$ are copies of $(g_1, \ldots, g_n)$, the ordered list $\mathcal{F}$ is empty. We apply the following operations successively on the data.

(i) **Elementary transformation :** Apply an elementary transformation, say R_i^ϵ with $1 \leq i \leq m-1$ and $\epsilon = \pm 1$, on the m-tuple $(h_1, \ldots, h_m)$ and the iterated m-tuple $(H_1, \ldots, H_m)$. Append (m, R_i^ϵ) to $\mathcal{F}$.

(ii) **Contraction :** For a pair of integers $1 \leq l < r \leq n$, we replace the tuple $(h_1, \ldots, h_m)$ with

$$(h_1, \ldots, h_{l-1}, h_l \cdots h_r, h_{r+1}, \ldots, h_m)$$

and replace the iterated tuple $(H_1, \ldots, H_m)$ with

$$(H_1, \ldots, H_{l-1}, (H_l, \ldots, H_r), H_{r+1}, \ldots, H_m).$$

Append (l, r) to $\mathcal{F}$.

(iii) **Restoration :** Take the last pair of the form (l, r) in $\mathcal{F}$, still denoted by (l, r). Let $\mathcal{F}'$ be the sub-list of $\mathcal{F}$ which consists of the elements after (l, r). Remove (l, r) and all the elements after (l, r) from $\mathcal{F}$.
Set $k = l$ and $m' = m + (r - l)$. We consider each pair $(\mu, \sigma) = (m, R_i^\epsilon)$ in $\mathcal{F}'$ with the order.
  - If $1 \leq i \leq k-2$, then append (m', R_i^ϵ) to $\mathcal{F}$.
  - If $k+1 \leq i \leq m$, then append $(m', R_{i+(r-l)}^\epsilon)$ to $\mathcal{F}$.
  - If $\sigma = R_{k-1}$, then append the pairs $(m', R_{k-1}), \ldots, (m', R_{k-1+(r-l)})$ to $\mathcal{F}$ and replace k with $k-1$.

    In this case, the elementary transformation σ acts on an iterated m-tuple of the form $(\mathcal{H}_1, \ldots, \mathcal{H}_{k-1}, (\mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}), \mathcal{H}_{k+1}, \ldots, \mathcal{H}_m)$ via

    $$(\ldots, \mathcal{H}_{k-1}, (\mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}), \ldots) \xrightarrow{R_{k-1}} (\ldots, (\mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}), \mathcal{H}_{k-1}^{ev(\mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1})}, \ldots).$$

    The new pairs $(m', R_{k-1}), \ldots, (m', R_{k-1+(r-l)})$ in $\mathcal{F}$ act on an iterated m'-tuple of the form $(\mathcal{H}_1, \ldots, \mathcal{H}_{k-1}, \mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}, \mathcal{H}_{k+1}, \ldots, \mathcal{H}_m)$ via

    $$\begin{aligned}(\ldots, \mathcal{H}_{k-1}, \mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}, \ldots) &\xrightarrow{R_{k-1}} (\ldots, \mathcal{H}'_1, \mathcal{H}_{k-1}^{ev(\mathcal{H}'_1)}, \mathcal{H}'_2 \ldots, \mathcal{H}'_{r-l+1}, \ldots) \\ &\xrightarrow{R_k} (\ldots, \mathcal{H}'_1, \mathcal{H}'_2, \mathcal{H}_{k-1}^{ev(\mathcal{H}'_1)ev(\mathcal{H}'_2)}, \mathcal{H}'_3 \ldots, \mathcal{H}'_{r-l+1}, \ldots) \\ &\to \ldots \\ &\xrightarrow{R_{k-1+(r-k)}} (\ldots, \mathcal{H}'_1, \ldots, \mathcal{H}'_{r-l+1}, \mathcal{H}_{k-1}^{ev(\mathcal{H}'_1) \cdots ev(\mathcal{H}'_{r-l+1})}, \ldots).\end{aligned}$$

  - If $\sigma = R_k$, then append the pairs $(m', R_{k+(r-l)}), \ldots, (m', R_k)$ to $\mathcal{F}$ and replace k with $k+1$.
  - If $\sigma = R_{k-1}^{-1}$, then append the pairs $(m', R_{k-1}^{-1}), \ldots, (m', R_{k-1+(r-l)}^{-1})$ to $\mathcal{F}$ and replace k with $k-1$.
  - If $\sigma = R_k^{-1}$, then append the pairs $(m', R_{k+(r-l)}^{-1}), \ldots, (m', R_k^{-1})$ to $\mathcal{F}$ and replace k with $k+1$.

Finally, suppose that $H_k = (H'_1, \ldots, H'_{r-l+1})$. We replace $(h_1, \ldots, h_m)$ with

$$(h_1, \ldots, h_{k-1}, ev(H'_1), \ldots, ev(H'_{r-l+1}), h_{k+1}, \ldots, h_m)$$

and replace $(H_1, \ldots, H_m)$ with

$$(H_1, \ldots, H_{k-1}, H'_1, \ldots, H'_{r-l+1}, H_{k+1}, \ldots, H_m).$$

Note that above operations can be applied in any order, possibly each appears many times and different operations may alternate with each other. However, we apply operations only finitely many times. The following lemma shows the main property of these operations.

Lemma 2.1.8. *If $m = n$, then $(h_1, \ldots, h_m)$ coincides with the resulting tuple of $(g_1, \ldots, g_n)$ after applying all elementary transformations σ occurring in $\mathcal{F}$ with the order.*

Proof. Let (l, r) be the last pair of integers in $\mathcal{F}$ which indicates the last contraction operation and replaces $h_l, \dots, h_r$ with $h_l \cdots h_r$. The product exactly corresponds to the k-th component of the m-tuple after each of the subsequent elementary transformations, where k is introduced in the restoration operation. Therefore, the restoration cancels the contraction and constructs the corresponding elementary transformations on the m'-tuple. We conclude the lemma by induction. $\square$

A direct application of the above operations requires us to maintain a lot of data, which would be a massive and tedious project. To simplify the application, our usage only focuses on the replacement

$$(h_1, \dots, h_m) \dashrightarrow (h_1, \dots, h_{l-1}, h_l \cdots h_r, h_{r+1}, \dots, h_m)$$

of the contraction; when applying the restoration, we enumerate all possible patterns of the corresponding contraction instead. Therefore, the iterated tuple $(H_1, \dots, H_m)$ and the ordered list $\mathcal{F}$ never appear in the argument.

More delicate operations for tuples in the modular group and their properties will be introduced in Proposition 2.3.10 and Proposition 2.3.29. We need the following definition in the sequel :

Definition 2.1.9. Let $(g_1, \dots, g_n)$ and $(h_1, \dots, h_m)$ be tuples in G with $n \geq m$. The tuple $(g_1, \dots, g_n)$ is said to be an $(h_1, \dots, h_m)$-*expansion* (or an *expansion* of $(h_1, \dots, h_m)$) if there exist integers $0 = i_0 < i_1 < i_2 < \dots < l_m = n$ such that $g_{i_{j-1}+1} \cdots g_{i_j} = h_j$ for each of $j = 1, \dots, m$.

Suppose that $(g_1, \dots, g_n)$ is an expansion of $(h_1, \dots, h_m)$. Then the associated contraction operations consist of m contractions that replace $(g_1, \dots, g_n)$ with $(h_1, \dots, h_m)$.

2.1.3 Direct sums of fibrations and their global monodromies

Recall that the *type* $\mathcal{O}(f)$ of singularities of a torus fibration f is a multi-set of fibre monodromies counted with multiplicity. Let $f_1 : M_1 \to S^2$, $f_2 : M_2 \to S^2$ be torus fibrations, possibly with different numbers of singular fibres. Let $f_1 \oplus f_2$ be a direct sum of f_1 and f_2. The global monodromy of $f_1 \oplus f_2$ depends on the fibre-connected sum $M_1 \oplus_\beta M_2$, the base point p on S^2 and the set of generators for the fundamental group $\pi_1(S^2 \setminus B)$. To be precise, a global monodromy of $f_1 \oplus f_2$ is a concatenation of two sub-tuples, say

$$(\psi_1^{-1}\phi_{1,1}\psi_1, \dots, \psi_1^{-1}\phi_{1,n_1}\psi_1) \bullet (\psi_2^{-1}\phi_{2,1}\psi_2, \dots, \psi_2^{-1}\phi_{2,n_2}\psi_2),$$

such that $(\phi_{1,1}, \dots, \phi_{1,n_1})$ and $(\phi_{2,1}, \dots, \phi_{2,n_2})$ are global monodromies of f_1 and f_2 respectively, $\psi_1, \psi_2 \in \mathrm{SL}(2,\mathbb{Z})$ and at least one of ψ_1, ψ_2 is 1. In general, global monodromies of different direct sums or of the same direct sum but with different base points are not Hurwitz equivalent.

For any n-tuple $(\phi_1, \dots, \phi_n)$ in $\mathrm{SL}(2,\mathbb{Z})$ with $\phi_1 \cdots \phi_n = 1$, we use $f_{(\phi_1,\dots,\phi_n)}$ to denote a torus fibration that has a global monodromy equal to $(\phi_1, \dots, \phi_n)$, if it exists. We use the notation $f^L_{(\phi_1,\dots,\phi_n)}$ for such a fibration that is also a Lefschetz fibration. Lemma 2.1.10 will point out that we can always work with such a Lefschetz fibration up to expansion. Let us first recall some facts about $\mathrm{SL}(2,\mathbb{Z})$ and Lefschetz fibrations.

Set $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Let $L = -ABA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $R = -AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$ have been described using the geometry of continued fractions (see [Ser85 ; Kar13 ; Mos16]). They are classified according to the trace, which is conjugacy invariant, as follows.

(0) For trace 0, there are two conjugacy classes represented by B and $-B$.

For nonzero trace, the conjugacy classes come in opposite pairs, represented by a matrix M and its opposite $-M$ with $tr(M) > 0$ and $tr(-M) < 0$.

(1) For trace 1, there are two conjugacy classes represented by A and $-A^2$.
For trace -1, there are two conjugacy classes represented by $-A$ and A^2.

(2) For trace 2, there is a $\mathbb{Z}$-indexed families of conjugacy classes represented by L^r with $r \in \mathbb{Z}$.
For trace -2, there is a $\mathbb{Z}$-indexed families of conjugacy classes represented by $-L^r$ with $r \in \mathbb{Z}$.

(3) For trace 3, there is only one conjugacy class represented by LR.
For trace -3, there is only one conjugacy class represented by $-LR$.

(≥ 3) In general, for trace of absolute value ≥ 3, the words of the form $\pm R^{j_1}L^{k_1}R^{j_2}L^{k_2}\cdots R^{j_m}L^{k_m}$ with $m \geq 1$, $j_1,\ldots,j_m,k_1,\ldots,k_m \geq 1$ represent all conjugacy classes. Conversely, different words of this form up to cyclic conjugacy belong to different conjugacy classes.

Recall that the fibre monodromies of torus Lefschetz fibrations are conjugates of L.
For convenience, we set $\check{L} = -A^2B$ and $\widehat{L} = -BA^2$, which are conjugates of L.

Lemma 2.1.10. *Let $(\phi_1,\ldots,\phi_n)$ be an n-tuple in $\mathrm{SL}(2,\mathbb{Z})$ with $\phi_1\cdots\phi_n = 1$. There exists a torus Lefschetz fibration f^L, one of whose global monodromy is an expansion of $(\phi_1,\ldots,\phi_n)$.*

Proof. It suffices to show that the semigroup generated by $\check{L}$ and $\widehat{L}$ is exactly $\mathrm{SL}(2,\mathbb{Z})$. It follows from that $\check{L}\widehat{L} = A$ and $\check{L}\widehat{L}\check{L} = B$ whose inverses are A^5 and B^3 respectively. □

Now we describe the following tuples with respect to a multi-set $\mathcal{O}$ of fibre monodromies. Their induced fibrations $f_{T_{\mathcal{O},0}}$, $f_{T_{\mathcal{O},1}}$, $f_{T_{\mathcal{O},2}}$ and $f_{T_{\mathcal{O}}}$ will stabilise torus fibrations.

Definition 2.1.11. Suppose that $\mathcal{O}$ is a multi-set of conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$.

1) We define $T_{\mathcal{O},0}$ as $(\check{L},\widehat{L},A^2,\check{L},\widehat{L},A^2)$.
2) We define $T_{\mathcal{O},1}$ as an empty tuple if there does not exist a conjugacy class of trace ± 3 in $\mathcal{O}$, otherwise
$$T_{\mathcal{O},1} = (B,B,B,B)\bullet(-A^2BAB, BA, -ABA)^3.$$
3) We define $T_{\mathcal{O},2}$ as the concatenation of the following tuples.
   i) If the conjugacy class represented by ϵL^r with $r \geq 2$ and $\epsilon \in \{1,-1\}$ occurs $m \geq 1$ times in $\mathcal{O}$, take m copies of
   $$\underbrace{(L,\ldots,L,L^{-r})}_{r+1 \text{ components}}.$$
   ii) If the conjugacy class represented by ϵR^2 with $r \geq 2$ and $\epsilon \in \{1,-1\}$ occurs $m \geq 1$ times in $\mathcal{O}$, take m copies of
   $$\underbrace{(R,\ldots,R,R^{-r})}_{r+1 \text{ components}}.$$
   iii) Suppose that a conjugacy class of elements with $|trace| \geq 4$ is represented by
   $$\epsilon R^{j_1}L^{k_1}R^{j_2}L^{k_2}\cdots R^{j_m}L^{k_m}$$
   with $\epsilon = \{1,-1\}$, $m \geq 1$, $j_1,\ldots,j_m$, $k_1,\ldots,k_m \geq 1$. If the conjugacy class occurs $m \geq 1$ times in $\mathcal{O}$, take m copies of
   $$\Big(\underbrace{R,\ldots,R}_{j_1 \text{ components}}, \overbrace{L,\ldots,L}^{k_1 \text{ components}}, \ldots, \underbrace{R,\ldots,R}_{j_m \text{ components}}, \overbrace{L,\ldots,L}^{k_m \text{ components}}, (R^{j_1}L^{k_1}\cdots R^{j_m}L^{k_m})^{-1}\Big).$$

Eventually, we define $T_{\mathcal{O}}$ as $T_{\mathcal{O},0}\bullet T_{\mathcal{O},1}\bullet T_{\mathcal{O},2}$.

2.1.4 Hurwitz equivalence of global monodromies

If two global monodromies of torus fibrations are Hurwitz equivalent, then they must have the same number of branch points and the same type of singularities. The following theorem shows that the global monodromies of torus fibrations with the same type of singularities become Hurwitz equivalent up to fibre sum stabilisations.

Theorem 2.1.12. *Given a torus fibration, let $\mathcal{O}$ be the type of singularities. Suppose that f_0 is one of the following :*

i) a torus fibration, one of whose global monodromy is $(h_1,\ldots,h_m) = T_{\mathcal{O}}$;
ii) a torus Lefschetz fibration, one of whose global monodromy $(h_1,\ldots,h_m)$ is a $T_{\mathcal{O}}$-expansion.

Then all global monodromies of all direct sums $f\oplus f_0$ are Hurwitz equivalent for all torus fibrations f with $\mathcal{O}(f) = \mathcal{O}$. Moreover, these global monodromies have a specific normal form determined by $\mathcal{O}$ and $(h_1,\ldots,h_m)$ as follows :

$$(g_1,\ldots,g_l)\bullet\prod_i(\phi_{i,1},\ldots,\phi_{i,n_i})$$

where $g_1\cdots g_l = I$, $(g_1,\ldots,g_l)$ is the sub-tuple of $(h_1,\ldots,h_m)$ either equal to $T_{\mathcal{O},0}$ or corresponding to $T_{\mathcal{O},0}$, $\phi_{i,1}\cdots\phi_{i,n_i} = \pm I$ for each i and each $(\phi_{i,1},\ldots,\phi_{i,n_i})$ is either

- *a tuple of the form (X,Y) with $XY = \pm I$, or*
- *a tuple of $\pm A$, $\pm A^2$, $\pm B$, $\pm\check{L}$, $\pm L$, $\pm\hat{L}$, $\pm\check{L}^{-1}$, $\pm L^{-1}$, $\pm\hat{L}^{-1}$, except for at most 1 component.*

Remark 2.1.13. Theorem 2.1.12 and Lemma 2.1.10 imply the main result Theorem A. Let $H_{12} = (\check{L},\hat{L})^6$ and

$$H_{60} = (\check{L},\hat{L})^6 \bullet (\check{L},\hat{L},\check{L})^4 \bullet (\check{L},L,\check{L},\hat{L},\check{L},\hat{L},\check{L},\hat{L},\check{L},\hat{L},L,L)^3$$

be tuples in $\mathrm{SL}(2,\mathbb{Z})$. Theorem B and C again follow from Theorem 2.1.12, where the torus Lefschetz fibrations f^L_{12} and f^L_{60} are $f^L_{H_{12}}$ and $f^L_{H_{60}}$ respectively.

Remark 2.1.14. The normal form given in Theorem 2.1.12, though its precise form is not given, satisfies a remarkable property, called *swappability*. We explain the swappability in Subsection 2.2.1 but as a consequence, we have Proposition 2.1.15.

Let $\iota : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$ be the natural group homomorphism.

Proposition 2.1.15. *For $i = 1,2$, let $\prod_j(\phi^{(i)}_{j,1},\ldots,\phi^{(i)}_{j,n_j})$ be a tuple in $\mathrm{SL}(2,\mathbb{Z})$ such that*

$$\phi^{(i)}_{j,1}\cdots\phi^{(i)}_{j,n_j} = \pm I$$

for each j and each sub-tuple $(\phi^{(i)}_{j,1},\ldots,\phi^{(i)}_{j,n_j})$ is either
- *a tuple of the form (X,Y) with $XY = \pm I$, or*
- *a tuple of $\pm A$, $\pm A^2$, $\pm B$, $\pm\check{L}$, $\pm L$, $\pm\hat{L}$, $\pm\check{L}^{-1}$, $\pm L^{-1}$, $\pm\hat{L}^{-1}$, except for at most 1 component.*

Let $(g_1,\ldots,g_l)$ be a tuple in $\mathrm{SL}(2,\mathbb{Z})$ that is either equal to $T_{\mathcal{O},0}$ or a $T_{\mathcal{O},0}$ expansion. Suppose that

$$[Cl(\phi^{(1)}_{j,1}),\ldots,Cl(\phi^{(1)}_{j,n_j}) \mid j] = [Cl(\phi^{(2)}_{j,1}),\ldots,Cl(\phi^{(2)}_{j,n_j}) \mid j]$$

and

$$\prod_j(\iota(\phi^{(1)}_{j,1}),\ldots,\iota(\phi^{(1)}_{j,n_j})) = \prod_j(\iota(\phi^{(2)}_{j,1}),\ldots,\iota(\phi^{(2)}_{j,n_j})).$$

Then,

$$(g_1,\ldots,g_l)\bullet\prod_j(\phi^{(1)}_{j,1},\ldots,\phi^{(1)}_{j,n_j}) \sim (g_1,\ldots,g_l)\bullet\prod_j(\phi^{(2)}_{j,1},\ldots,\phi^{(2)}_{j,n_j}).$$

We need a deeper understanding of tuples in $\mathrm{PSL}(2,\mathbb{Z})$. Set $a = \iota(A)$, $b = \iota(B)$. Recall that $\mathrm{PSL}(2,\mathbb{Z})$ is generated by a and b with the relation $a^3 = b^2 = 1$. Some other elements are marked as follows :

$$s_0 = a^2b, s_1 = aba, s_2 = ba^2, t_0 = ba, t_1 = a^2ba^2, t_2 = ab$$

Here $\iota(L) = s_1$, $\iota(R) = t_2$, $\iota(\check{L}) = s_0$ and $\iota(\hat{L}) = s_2$. We further emphasise that $s_it_i = 1$ for $i = 0,1,2$. Elements s_0, s_1, s_2 are conjugate to each other and t_0, t_1, t_2 are conjugate to each other.

Elements in $\mathcal{S} = \{a,a^2,b,s_0,s_1,s_2,t_0,t_1,t_2\}$ are called "short" and elements in

$$\mathcal{S}_2 = \mathcal{S}\cup\{bab, ba^2b, a^2ba, aba^2, a^2bab, ababa, baba^2, ba^2ba, a^2ba^2ba^2, aba^2b\}$$

are called "almost short".

The following improves and extends Theorem 3.6 in [Mat85], which divides the tuples of elements in $\mathrm{PSL}(2,\mathbb{Z})$ conjugate to a, a^2, b, s_0 or t_0 into two categories and, for tuples in the second category, presents the normal forms.

Theorem 2.1.16. *Let $g_1,\ldots,g_n \in \mathrm{PSL}(2,\mathbb{Z})$ be conjugates of a, a^2, b, aba or a^2ba^2 satisfying $g_1\cdots g_n = 1$. Suppose that p_a of them are conjugates of a, q_a of them are conjugates of a^2, n_b of them are conjugates of b, p of them are conjugates of s_0 and q of them are conjugates of t_0 with p_a, q_a, n_b, p, $q \geq 0$ and $p_a+q_a+n_b+p+q = n$. Then,*

1. *if $p = q$, $|p_a - q_a| \equiv 0 \pmod 3$ and n_b is even, then the n-tuple $(g_1,\ldots,g_n)$ is Hurwitz equivalenet to*
$$(k_1,k_1^{-1},\ldots,k_{n'},k_{n'}^{-1},l_1,l_1,l_1,\ldots,l_{n''},l_{n''},l_{n''})$$
with $n'+n'' = n$, $k_i,l_j \in G$ and $l_j^3 = 1$, $i = 1,\ldots,n'$, $j = 1,\ldots,n''$;
2. *otherwise, the n-tuple $(g_1,\ldots,g_n)$ is Hurwitz equivalent to the concatenation of*

$$(s_0,s_2,s_0,s_2,s_0,s_2)^{\lfloor\max\{p-q,0\}/6\rfloor}\bullet(t_0,t_2,t_0,t_2,t_0,t_2)^{\lfloor\max\{q-p,0\}/6\rfloor}\bullet(s_0,t_0)^{\min\{p,q\}}\bullet$$
$$(a,a^2)^{\min\{p_a,q_a\}}\bullet(b,b)^{\lfloor n_b/2\rfloor}\bullet(a,a,a)^{\lfloor\max\{p_a-q_a,0\}/3\rfloor}\bullet(a^2,a^2,a^2)^{\lfloor\max\{q_a-p_a,0\}/3\rfloor}$$

and at most one of the following tuples :

$$(a^2, s_0, s_2), (a, t_2, t_0), (a, s_0, s_0, s_2, s_0), (a^2, t_0, t_2, t_0, t_0), (b, s_0, s_2, s_0), (b, t_0, t_2, t_0),$$
$$(a, b, s_2), (a^2, b, t_0), (a, t_2, t_0, b, t_0, t_2, t_0), (a^2, s_0, s_2, b, s_0, s_2, s_0),$$
$$(a, a, s_0, s_2), (a^2, a^2, t_2, t_0), (a^2, a^2, s_0, s_0, s_2, s_0), (a, a, t_0, t_2, t_0, t_0),$$
$$(a^2, a^2, b, s_2), (a, a, b, t_0), (a^2, a^2, t_2, t_0, b, t_0, t_2, t_0), (a, a, s_0, s_2, b, s_0, s_2, s_0).$$

The resulting n-tuple is called the normal form of $(g_1, \dots, g_n)$

As a supplement, we have Theorem 2.1.17 and its modification.

Theorem 2.1.17. *Let $g_1, \dots, g_n \in \mathrm{PSL}(2, \mathbb{Z})$ be conjugates of a, a^2, b, s_0, t_0 or ababa satisfying $g_1 \cdots g_n = 1$. Suppose that m of them are conjugates of ababa. Take*

$$\mathcal{F}_{13} = (b, b, b, b, a^2bab, t_0, s_1, a^2bab, t_0, s_1, a^2bab, t_0, s_1).$$

Then $(g_1, \dots, g_n) \bullet \mathcal{F}_{13}$ is Hurwitz equivalent to

$$(h_1, \dots, h_{n-m-3-2\mu}) \bullet (a^2bab, ba^2ba)^{(m+3-\mu)/2} \bullet (a^2bab, t_0, s_1)^{\mu}$$

where
- *each component of $(h_1, \dots, h_{n-m-3-2\mu})$ is conjugate to one of a, a^2, b, s_0, t_0 ;*
- *$\mu = 3 - m$ if $m \leq 3$ and $\mu = (m+1) \mod 2$ otherwise.*

Theorem 2.1.18 (A modification of Theorem 2.1.17)**.** *Let $g_1, \dots, g_n \in \mathrm{PSL}(2, \mathbb{Z})$ be conjugates of a, a^2, b, s_0, t_0 or ababa satisfying $g_1 \cdots g_n = 1$. Suppose that m of them are conjugates of ababa. Take*

$$\mathcal{F}_{13} = (b, b, b, b, a^2bab, t_0, s_1, a^2bab, t_0, s_1, a^2bab, t_0, s_1).$$

Let $\mathcal{F}^L$ be an $\mathcal{F}_{13}$-expansion of conjugates of s_0, written as

$$(u_{1,1}, \dots, u_{1,k_1}, u_{2,1}, \dots, u_{2,k_2}, \dots, u_{13,1}, \dots, u_{13,k_{13}}),$$

such that $u_{i,1} \cdots u_{i,k_i}$ is equal to the i-th component of $\mathcal{F}_{13}$ for each of $i = 1, \dots, 13$. Then $(g_1, \dots, g_n) \bullet \mathcal{F}^L$ is Hurwitz equivalent to

$$(h_1, \dots, h_{n'}) \bullet (a^2bab, ba^2ba)^{(m-3+\mu)/2} \bullet \prod_{i=1}^{3-\mu} (u_{3i+2,1}, \dots, u_{3i+2,k_{3i+2}}, ba^2ba)$$
$$\bullet \prod_{i=3-\mu+1}^{3} (u_{3i+2,1}, \dots, u_{3i+2,k_{3i+2}}, u_{3i+3,1}, \dots, u_{3i+3,k_{3i+3}}, u_{3i+4,1}, \dots, u_{3i+4,k_{3i+4}})$$

where :
- *each component of $(h_1, \dots, h_{n'})$ is conjugate to one of a, a^2, b, s_0, t_0 ;*
- *$\mu = 3 - m$ if $m \leq 3$ and $\mu = (m+1) \mod 2$ otherwise.*

The proof of Theorem 2.1.12 relies on Theorem 2.1.16, Theorem 2.1.17 and the above modification of Theorem 2.1.17. We will prove Theorem 2.1.16, 2.1.17 and 2.1.18 in Section 2.3.

Proof of Theorem 2.1.12. Suppose that $(h_1, \dots, h_m) = T_{\mathcal{O}}$ if f_0 is as in *i*), or $(h_1, \dots, h_m)$ is a $T_{\mathcal{O}}$-expansion if f_0 is as in *ii*), which is a global monodromy of f_0. We write it as a concatenation either
- of all the following tuples, or
- of the following tuples labelled (1), (2) and (4).

The list of tuples is as follows.
1. The tuple $(h_{1,1}, \dots, h_{1,m_1})$ is either $(\check{L}, \widehat{L}, A^2)$ or a $(\check{L}, \widehat{L}, A^2)$-expansion of conjugates of L.
2. The tuple $(h_{2,1}, \dots, h_{2,m_2})$ is either $(\check{L}, \widehat{L}, A^2)$ or a $(\check{L}, \widehat{L}, A^2)$-expansion of conjugates of L, which may be different from $(h_{1,1}, \dots, h_{1,m_1})$.
3. The tuple $(h_{3,1}, \dots, h_{3,m_3})$ is either $T_{\mathcal{O},1}$ or a $T_{\mathcal{O},1}$-expansion of conjugates of L.
4. The tuple $(h_{4,1}, \dots, h_{4,m_4})$ is either $T_{\mathcal{O},2}$ or a $T_{\mathcal{O},2}$-expansion of conjugates of L.

Step 1. We first show that any global monodromy of a direct sum $f \oplus f_0$ can be transformed into

$$(\phi_1, \ldots, \phi_n) \bullet (h_1, \ldots, h_m)$$

where $(\phi_1, \ldots, \phi_n)$ is a global monodromy of f.

Given a base point p of $f \oplus f_0$, a global monodromy of the direct sum with respect to p is the concatenation

$$(\psi_1^{-1}\phi_1\psi_1, \ldots, \psi_1^{-1}\phi_n\psi_1) \bullet \prod_{i \in \mathcal{I}} (\psi_2^{-1}h_{i,1}\psi_2, \ldots, \psi_2^{-1}h_{i,m_i}\psi_2)$$

with the tuple $(\phi_1, \ldots, \phi_n)$ a global monodromy of f, elements ψ_1, $\psi_2 \in \mathrm{SL}(2, \mathbb{Z})$ and the index set $\mathcal{I}$ either $\{1,2,3,4\}$ or $\{1,2,4\}$. Each tuple in the concatenation has the product of components equal to $\pm I$ and both of the tuples $(\psi_2^{-1}h_{1,1}\psi_2, \ldots, \psi_2^{-1}h_{1,m_1}\psi_2)$ and $(\psi_2^{-1}h_{2,1}\psi_2, \ldots, \psi_2^{-1}h_{2,m_2}\psi_2)$ contain generating sets. By Lemma 2.1.6, we can eliminate all the ψ_1, ψ_2 in the global monodromy using elementary transformations. Rewrite the resulting tuple as

$$(\phi_1, \ldots, \phi_n) \bullet (g_1, \ldots, g_l) \bullet (h_{3,1}, \ldots, h_{3,m_3})^{[3 \in \mathcal{I}]} \bullet (h_{4,1}, \ldots, h_{4,m_4})$$

where $[3 \in \mathcal{I}] = 1$ if $3 \in \mathcal{I}$ and $[3 \in \mathcal{I}] = 0$ if $3 \notin \mathcal{I}$, such that $(g_1, \ldots, g_l)$ is either $T_{\mathcal{O},0}$ or a $T_{\mathcal{O},0}$-expansion of conjugates of L.

Step 2. We show that the above resulting tuple is Hurwitz equivalent to

$$(\varphi_1, \ldots, \varphi_{n'}) \bullet (g_1, \ldots, g_l) \bullet (h_{3,1}, \ldots, h_{3,m_3})^{[3 \in \mathcal{I}]} \bullet (h'_{4,1}, \ldots, h'_{4,m_4})$$

such that

- $(\varphi_1, \ldots, \varphi_{n'})$ is a tuple of elements in simple conjugacy classes such that $\varphi_1 \cdots \varphi_{n'} = \pm I$;
- $(h'_{4,1}, \ldots, h'_{4,m_4})$ depends only on $(h_{4,1}, \ldots, h_{4,m_4})$ and components of $(\phi_1, \ldots, \phi_n)$ in non-simple conjugacy classes.

If $(h_{4,1}, \ldots, h_{4,m_4})$ is a $T_{\mathcal{O},2}$-expansion, then using contractions on $(h_{4,1}, \ldots, h_{4,m_4})$ as in Subsection 2.1.2 we replace $(h_{4,1}, \ldots, h_{4,m_4})$ with $T_{\mathcal{O},2}$. The definition of $T_{\mathcal{O},2}$ states that it is the concatenation of several sub-tuples. These sub-tuples are in one-to-one correspondence with the singular fibres of f whose fibre monodromies belong to non-simple conjugacy classes and they are further in one-to-one correspondence with the components of $(\phi_1, \ldots, \phi_n)$ excluding those of trace 0, ± 1, ± 3 or conjugate to $\pm L$, $\pm R$.

Suppose that there exists an $(r+1)$-sub-tuple of the form $(L, \ldots, L, L^{-r})$ in $T_{\mathcal{O},2}$ with $r \geq 2$. We take the corresponding component, say ϕ_i, which is equal to $\epsilon h^{-1}L^r h$ with $\epsilon = \pm 1$ and $h \in \mathrm{SL}(2, \mathbb{Z})$. Since $(g_1, \ldots, g_l)$ contains a generating set, by Lemma 2.1.6, we replace the $(r+1)$-sub-tuple with

$$(h^{-1}Lh, \ldots, h^{-1}Lh, h^{-1}L^{-r}h).$$

By Lemma 2.1.3, we further replace ϕ_i with $(h^{-1}Lh, \ldots, h^{-1}Lh)$ and replace the above $(r+1)$-sub-tuple with $(\epsilon h^{-1}L^r h, h^{-1}L^{-r}h)$. Again by Lemma 2.1.6, the pair $(\epsilon h^{-1}L^r h, h^{-1}L^{-r}h)$ can be transformed into $(\epsilon L^r, L^r)$.

We have similar arguments for sub-tuples of the form $(R, \ldots, R, R^{-r})$ or of the form

$$(R, \ldots, R, L, \ldots, L, \ldots, R, \ldots, R, L, \ldots, L, (R \ldots RL \ldots L \ldots R \ldots RL \ldots L)^{-1})$$

as in Definition 2.1.11. For the restoration, according to each component in $T_{\mathcal{O},2}$, we rewrite the corresponding component as a sub-tuple. Notice that if such a component belongs to some simple conjugacy class, then it is replaced by a sub-tuple of conjugates of L. Hence the resulting tuple is as desired.

We will not modify $(h'_{4,1}, \ldots, h'_{4,m_4})$ anymore.

Step 3. Suppose that n_+ components of $(\varphi_1, \ldots, \varphi_{n'})$ are of trace 3 and n_- components of $(\varphi_1, \ldots, \varphi_{n'})$ are of trace -3. If $n_+ + n_- = 0$, then take $(\varphi'_1, \ldots, \varphi'_{n''}) = (\varphi_1, \ldots, \varphi_{n'})$ and skip the step. Otherwise, $[3 \in \mathcal{I}] = 1$. We further show that, by elementary transformations, $(\varphi_1, \ldots, \varphi_{n'}) \bullet (g_1, \ldots, g_l) \bullet (h_{3,1}, \ldots, h_{3,m_3})$ can be transformed into

$$(\varphi'_1, \ldots, \varphi'_{n''}) \bullet (g_1, \ldots, g_l) \bullet (\bar{h}_1, \ldots, \bar{h}_{\bar{m}})$$

such that

- $(\varphi'_1, \ldots, \varphi'_{n''})$ is a tuple of elements either of trace $0, \pm 1$ or conjugate to $\pm L$ or $\pm R$,
- $(\bar{h}_1, \ldots, \bar{h}_{\bar{m}})$ depends only on n_+, n_- and $(h_{3,1}, \ldots, h_{3,m_3})$.

If $(h_{3,1}, \dots, h_{3,m_3})$ is an expansion of $T_{\mathcal{O},1}$, then using contractions on $(h_{3,1}, \dots, h_{3,m_3})$ as in Subsection 2.1.2 we replace it with $T_{\mathcal{O},1}$. By applying Theorem 2.1.17 to $(\iota(\varphi_1), \dots, \iota(\varphi_{n'})) \bullet \mathcal{F}_{13}$, the tuple is transformed into

$$(\varphi'_1, \dots, \varphi'_{n''}) \bullet (g_1, \dots, g_l) \bullet \prod_{i=1}^{k} (\psi_{i,0}, \psi_{i,1}) \bullet (-A^2BAB, BA, -ABA)^{\mu}$$

such that each of $\iota(\varphi'_i)$, $i = 1, \dots, n''$ is conjugate to a, a^2, b, s_0 or t_0, $\iota(\psi_{i,0}) = a^2bab$, $\iota(\psi_{i,1}) = ba^2ba$ and $\mu \leq 3$. The number μ is determined by $n_+ + n_-$, which further separates the cases.

Then, the restoration operations apply on the tuple. Some components of $(\varphi'_1, \dots, \varphi'_{n''})$ are replaced by sub-tuples of elements conjugate to L, while keeping each component conjugate to some preimage of a, a^2, b, s_0 or t_0. The remaining components that might be modified by the restoration are exactly the components of the last μ sub-triples and the last $3 - \mu$ components denoted by $\psi_{i,0}$. By Theorem 2.1.18 they are replaced by certain sub-tuples of $(h_{3,1}, \dots, h_{3,m_3})$.

The remaining components of $\prod_{i=1}^{k} (\psi_{i,0}, \psi_{i,1})$ are of trace ± 3 and they are either $\pm A^2BAB$ or $\pm BA^2BA$. To restrict their dependencies only on n_+ and n_-, we have to show that their signs can be rearranged to certain positions, but this follows from Proposition 2.1.15.

Step 4. We conclude the proof of Theorem 2.1.12 by showing that $(\varphi'_1, \dots, \varphi'_{n''}) \bullet (g_1, \dots, g_l)$ is Hurwitz equivalent to $(\varphi''_1, \dots, \varphi''_{n''}) \bullet (g_1, \dots, g_l)$ such that $(\varphi''_1, \dots, \varphi''_{n''})$ depends only on the multi-set $\mathcal{O}$.

Applying Theorem 2.1.16 to $(\iota(\varphi'_1), \dots, \iota(\varphi'_{n''}))$, we transform the tuple $(\varphi'_1, \dots, \varphi'_{n''})$ into a new tuple, denoted by $(\varphi''_1, \dots, \varphi''_{n''})$. For the first case in Theorem 2.1.16, as $(g_1, \dots, g_l)$ contains a generating set, applying Lemma 2.1.6 we further transform the concatenation into a resulting tuple, denoted by $(\varphi''_1, \dots, \varphi''_{n''}) \bullet (g_1, \dots, g_l)$, satisfying

$$(\iota(\varphi''_1), \dots, \iota(\varphi''_{n''})) = (s_0, t_0)^{\mu_1} \bullet (a, a^2)^{\mu_2} \bullet (b, b)^{\mu_3} \bullet (a, a, a)^{\mu_4} \bullet (a^2, a^2, a^2)^{\mu_5}$$

with $\mu_1, \dots, \mu_5$ determined by $\mathcal{O}$. The theorem follows from Proposition 2.1.15. □

Remark 2.1.19. Alternatively, instead of using Proposition 2.1.15, one may apply the substitutions

$$\begin{aligned}(\epsilon_1 A^2BAB, \epsilon_2 BA^2BA) \bullet (g_1, \dots, g_l) &\longrightarrow (\epsilon_1 BA^2BA, \epsilon_2 B^2A^2BAB^{-1}) \bullet (g_1, \dots, g_l) \\ = (\epsilon_1 BA^2BA, \epsilon_2 A^2BAB) \bullet (g_1, \dots, g_l) &\longrightarrow (\epsilon_2 A^2BAB, \epsilon_1 BA^2BA) \bullet (g_1, \dots, g_l).\end{aligned}$$

at the end of Step 3 and

$$(\tau_1 A^2B, \tau_2 BA^2, \tau_3 A^2B) \longleftrightarrow (\tau_2 BA^2, \tau_1 ABA, \tau_3 A^2B) \longleftrightarrow (\tau_2 BA^2, \tau_3 A^2B, \tau_1 BA^2).$$

at the end of Step 4, where ϵ_1, ϵ_2, τ_1, τ_2, $\tau_3 \in \{-I, +I\}$ are arbitrary. They appeared in an earlier version of this paper.

We end with the proof of Corollary A.

Proof of Corollary A. The fibre monodromy of a singular fibre distinguishes the type in the Kodaira classification. The corollary follows from Theorem B. □

2.2 Swappability and local models

2.2.1 Swappability of the normal form

This subsection introduces the notion of *swappability* for a tuple in $\mathrm{SL}(2, \mathbb{Z})$ with a stabilisation.

Definition 2.2.1. Let $(\phi_1, \dots, \phi_n)$ and $(g_1, \dots, g_l)$ be tuples in $\mathrm{SL}(2, \mathbb{Z})$. Suppose that, for any abelian group G and tuples $(\epsilon_1, \dots, \epsilon_n)$, $(\epsilon'_1, \dots, \epsilon'_n)$, $(\sigma_1, \dots, \sigma_l)$ in G such that the following multi-sets of conjugacy classes in $G \times \mathrm{SL}(2, \mathbb{Z})$ coincide :

$$[Cl\big((\epsilon_1, \phi_1)\big), \dots, Cl\big((\epsilon_n, \phi_n)\big)] = [Cl\big((\epsilon'_1, \phi_1)\big), \dots, Cl\big((\epsilon'_n, \phi_n)\big)],$$

we have

$$\big((\sigma_1, g_1), \dots, (\sigma_l, g_l)\big) \bullet \big((\epsilon_1, \phi_i), \dots, (\epsilon_n, \phi_n)\big) \sim \big((\sigma_1, g_1), \dots, (\sigma_l, g_l)\big) \bullet \big((\epsilon'_1, \phi_i), \dots, (\epsilon'_n, \phi_n)\big).$$

In this case, we say that $(\phi_1, \dots, \phi_n)$ is $(g_1, \dots, g_l)$-stabilised *swappable*.

Remark 2.2.2. The normal form given in Theorem 2.1.12 is an example of swappable tuples, which is guaranteed by Proposition 2.2.3.

Proposition 2.2.3. *Let $\prod_i(\phi_{i,1},\ldots,\phi_{i,n_i})$ be a tuple in* $\mathrm{SL}(2,\mathbb{Z})$ *such that $\phi_{i,1}\cdots\phi_{i,n_i}=\pm I$ for each i and each $(\phi_{i,1},\ldots,\phi_{i,n_i})$ is either*

- *a tuple of the form (X,Y) with $XY=\pm I$, or*
- *a tuple of $\pm A$, $\pm A^2$, $\pm B$, $\pm\check{L}$, $\pm L$, $\pm\widehat{L}$, $\pm\check{L}^{-1}$, $\pm L^{-1}$, $\pm\widehat{L}^{-1}$, except for at most 1 component.*

Let $(g_1,\ldots,g_l)$ be a tuple in $\mathrm{SL}(2,\mathbb{Z})$ *either equal to $T_{\mathcal{O},0}$ or a $T_{\mathcal{O},0}$ expansion. Then the tuple $\prod_i(\phi_{i,1},\ldots,\phi_{i,n_i})$ is $(g_1,\ldots,g_l)$-stabilised swappable.*

Proof. We need only prove the proposition for the case $(g_1,\ldots,g_l)=T_{\mathcal{O},0}$.

Set $(\phi_1,\ldots,\phi_n)=\prod_i(\phi_{i,1},\ldots,\phi_{i,n_i})$ and consider the tuple

$$\big((\sigma_1,g_1),\ldots,(\sigma_l,g_l)\big)\bullet\big((\epsilon_1,\phi_i),\ldots,(\epsilon_n,\phi_n)\big)$$

in $G\times\mathrm{SL}(2,\mathbb{Z})$. It suffices to show that, for any two components (ϵ_i,ϕ_i) and (ϵ_j,ϕ_j) such that ϕ_i is conjugate to ϕ_j, one can interchange ϵ_i and ϵ_j using elementary transformations.

When $\phi_i=\phi_j$, the swapping follows from Lemma 2.1.4.

When $\phi_i\neq\phi_j$ but ϕ_i, ϕ_j belong to different sub-tuples, using Lemma 2.1.6 for the sub-tuple containing (ϵ_i,ϕ_i), we transform the component (ϵ_i,ϕ_i) into (ϵ_i,ϕ_j). After swapping (ϵ_i,ϕ_j) and (ϵ_j,ϕ_j), we apply Lemma 2.1.6 again to make other components unchanged.

When $\phi_1\neq\phi_j$ and ϕ_i, ϕ_j belong to a sub-tuple not of the form (X,Y) with $XY=\pm I$, we must have ϕ_i conjugate to one of L, $-L$, L^{-1} and $-L^{-1}$. Recall $T_{\mathcal{O},0}=(\check{L},\widehat{L},A^2)\bullet(\check{L},\widehat{L},A^2)$ and notice that the (first) $(\check{L},\widehat{L},A^2)$ contains generating sets. If ϕ_i and ϕ_j are conjugate to $\pm L$, then take Q_i and Q_j be such that $Q_i^{-1}\phi_iQ_i=\pm L$ and $Q_j^{-1}\phi_jQ_j=\pm L$, therefore $Q_jQ_i^{-1}\phi_iQ_iQ_j^{-1}=\phi_j$ and $Q_iQ_j^{-1}\phi_jQ_jQ_i^{-1}=\phi_i$. Using Lemma 2.1.6 and Lemma 2.1.4, we have the following substitutions

$$\begin{aligned}
&(\ldots,(\epsilon_i,\phi_i),\ldots,(\epsilon_j,\phi_j),\ldots)\bullet((\delta_1,\check{L}),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\epsilon_i,Q_i^{-1}\phi_iQ_i),\ldots,(\epsilon_j,Q_i^{-1}\phi_jQ_i),\ldots)\bullet((\delta_1,\check{L}),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,\check{L}),\ldots,(\epsilon_j,Q_i^{-1}\phi_jQ_i),\ldots)\bullet((\epsilon_i,Q_i^{-1}\phi_iQ_i),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,Q_i\check{L}Q_i^{-1}),\ldots,(\epsilon_j,\phi_j),\ldots)\bullet((\epsilon_i,Q_i^{-1}\phi_iQ_i),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,Q_j^{-1}Q_i\check{L}Q_i^{-1}Q_j),\ldots,(\epsilon_j,Q_j^{-1}\phi_jQ_j),\ldots)\bullet((\epsilon_i,Q_i^{-1}\phi_iQ_i),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,Q_j^{-1}Q_i\check{L}Q_i^{-1}Q_j),\ldots,(\epsilon_i,Q_i^{-1}\phi_iQ_i),\ldots)\bullet((\epsilon_j,Q_j^{-1}\phi_jQ_j),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,Q_i\check{L}Q_i^{-1}),\ldots,(\epsilon_i,Q_jQ_i^{-1}\phi_iQ_iQ_j^{-1}),\ldots)\bullet((\epsilon_j,Q_j^{-1}\phi_jQ_j),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\delta_1,\check{L}),\ldots,(\epsilon_i,Q_i^{-1}Q_jQ_i^{-1}\phi_iQ_iQ_j^{-1}Q_i),\ldots)\bullet((\epsilon_j,Q_j^{-1}\phi_jQ_j),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\epsilon_j,Q_j^{-1}\phi_jQ_j),\ldots,(\epsilon_i,Q_i^{-1}Q_jQ_i^{-1}\phi_iQ_iQ_j^{-1}Q_i),\ldots)\bullet((\delta_1,\check{L}),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
\longrightarrow&(\ldots,(\epsilon_j,Q_iQ_j^{-1}\phi_jQ_jQ_i^{-1}),\ldots,(\epsilon_i,Q_jQ_i^{-1}\phi_iQ_iQ_j^{-1}),\ldots)\bullet((\delta_1,\check{L}),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2))\\
=&(\ldots,(\epsilon_j,\phi_i),\ldots,(\epsilon_i,\phi_j),\ldots)\bullet((\delta_1,\check{L}),(\delta_2,\widehat{L}),\ldots,(\delta_6,A^2)).
\end{aligned}$$

If ϕ_i and ϕ_j are conjugate to $\pm L^{-1}$, then using the contraction on $((\delta_2,\widehat{L}),(\delta_3,A^2))$ we have a similar sequence of substitutions.

When $\phi_i\neq\phi_j$ and ϕ_i, ϕ_j form a sub-tuple of the form (X,Y) with $XY=\pm I$, there exists $Q\in\mathrm{SL}(2,\mathbb{Z})$ such that $Q^{-1}\phi_iQ=\phi_j$ and therefore $Q^{-1}\phi_jQ=\phi_i$. By Lemma 2.1.6, the sub-tuple $((\epsilon_i,\phi_i),(\epsilon_j,\phi_j))$ can be transformed into $((\epsilon_i,Q^{-1}\phi_iQ),(\epsilon_j,Q^{-1}\phi_jQ))$. □

As a consequence, we prove Proposition 2.1.15.

Proof of Proposition 2.1.15. Let $G=\{1,-1\}$ be the group under multiplication. We define

$$\mathrm{SL}(2,\mathbb{Z})\ni\phi\mapsto\sharp(\phi)=(\epsilon,\psi)\in G\times\mathrm{SL}(2,\mathbb{Z})$$

such that $\mathrm{trace}(\psi)\geq 0$, $\epsilon=\mathrm{sgn}(\mathrm{trace}(\phi))$ if $\mathrm{trace}(\phi)\neq 0$ and $\epsilon=1$ otherwise. This map is well-defined, injective and conjugacy-preserving, but not a group homomorphism.

Consider the tuple

$$\prod_j((\epsilon_{j,1}^{(1)},\psi_{j,1}^{(1)}),\ldots,(\epsilon_{j,n_j}^{(1)},\psi_{j,n_j}^{(1)}))=\prod_j(\sharp(\phi_{j,1}^{(1)}),\ldots,\sharp(\phi_{j,n_j}^{(1)}))$$

and the tuple

$$\prod_j((\epsilon_{j,1}^{(2)},\psi_{j,1}^{(2)}),\ldots,(\epsilon_{j,n_j}^{(2)},\psi_{j,n_j}^{(2)}))=\prod_j(\sharp(\phi_{j,1}^{(2)}),\ldots,\sharp(\phi_{j,n_j}^{(2)}))$$

in $\times\,\mathrm{SL}(2,\mathbb{Z})$. Their components present the same conjugacy classes counted with multiplicity and $\psi_{j,k}^{(1)}=\psi_{j,k}^{(2)}$ for all j and k. Besides, each sub-tuple of $\prod_j(\psi_{j,1}^{(1)},\ldots,\psi_{j,n_j}^{(1)})$ is either a tuple of the form (X,Y) with $XY=\pm I$ or a tuple of A, $-A^2$, $\pm B$, $\check{L}$, L, $\widehat{L}$, $\check{L}^{-1}$, L^{-1} and $\widehat{L}^{-1}$. This proposition follows from Proposition 2.2.3. $\square$

2.2.2 Fibre-preserving homeomorphisms : from local to global

This subsection investigates fibre-preserving homeomorphisms between torus fibrations. We start with the following definitions.

Definition 2.2.4. Suppose that $f:M\to S^2$ is a torus fibration and $p_j\in S^2$ is a branch point. The singular fibre $f^{-1}(p_j)$ may be *locally symmetric* in the following sense. Let $U\subset S^2$ be any sufficiently small neighbourhood of p_j and $p\in\partial U$ be an arbitrary point. Identifying $f^{-1}(p)$ with $\mathbb{T}^2$, we use $\phi_j\in\mathrm{Mod}(\mathbb{T}^2)$ to denote the monodromy along ∂U at p. Let $\psi\in\mathrm{Mod}(\mathbb{T}^2)$ be an arbitrary mapping class class such that $\psi\phi_j=\phi_j\psi$. We suppose that there exists a (self-)homeomorphism $\Psi_M:f^{-1}(U)\to f^{-1}(U)$ such that $f\circ\Psi_M=f$ and $\Psi_M\mid_{f^{-1}(p)}$ represents ψ. In this case, we say that the singular fibre $f^{-1}(p)$ is *locally symmetric.*

In particular, all singular fibres of a torus Lefschetz fibration are locally symmetric.

Definition 2.2.5. Suppose that $f_1:M_1\to S^2$ and $f_2:M_2\to S^2$ are torus fibrations with branch sets $\mathcal{B}_1=\{p_j^{(1)}\}$ and $\mathcal{B}_2=\{p_j^{(2)}\}$. We say that the singular fibres $f_1^{-1}(p_j^{(1)})$ and $f_2^{-1}(p_j^{(2)})$ are *locally fibre-preserving homeomorphic* if, for any sufficiently small neighbourhood $U_j^{(1)}$ of $p_j^{(1)}$, there exist homeomorphisms $\Psi_{S,j}:U_j^{(1)}\to S^2$ and $\Psi_{M,j}:f_1^{-1}(U_j^{(1)})\to f_2^{-1}(\Psi_{S,j}(U_j^{(1)}))$ such that $f_2\circ\Psi_{M,j}=\Psi_{S,j}\circ f_1$. We further say that (M_1,f_1) and (M_2,f_2) are *fibre-preserving homeomorphic* if there exist homeomorphisms $\Psi_S:S^2\to S^2$ and $\Psi_M:M_1\to M_2$ such that $f_2\circ\Psi_M=\Psi_S\circ f_1$.

Definition 2.2.6. A locally fibre-preserving homeomorphism $(\Psi_{S,j},\Psi_{M,j})$ as above may be *compatible* with given global monodromies $(\phi_1^{(1)},\ldots,\phi_n^{(1)})$ of f_1 and $(\phi_1^{(2)},\ldots,\phi_n^{(2)})$ of f_2, in the following sense. Recall that the global monodromy is determined by a base point $p^{(i)}$ and a collection of loops $\gamma_1^{(i)},\ldots,\gamma_n^{(i)}$ based at $p^{(i)}$ such that $\gamma_j^{(i)}$ is exactly the boundary of a neighbourhood of $p_j^{(i)}$, say $\gamma_j^{(i)}=\partial D_j^{(i)}$, for $i=1,2$. Without loss of generality, assume that $U_j^{(1)}\subset D_j^{(1)}$ and $\Psi_{S,j}(U_j^{(1)})\subset D_j^{(2)}$. Let $\beta_j^{(i)}$ be an arbitrary path in $D_j^{(i)}$ joining $p^{(i)}$ to some point on $\partial D_j^{(i)}$, for $i=1,2$. The locally fibre-preserving homeomorphism $(\Psi_{M,j},\Psi_{S,j})$ is pushed forward to a homeomorphism $\psi\in\mathrm{Homeo}(\mathbb{T}^2)$ between the generic fibres at base points. We say that $(\Psi_{M,j},\Psi_{S,j})$ is *compatible* with the global monodromies if $[\psi]\phi_j^{(1)}=\phi_j^{(2)}[\psi]$.

The compatibility does not depend on the choice of $\beta_j^{(i)}$, for $i=1,2$. Indeed, for a different choice of $(\beta_j^{(1)},\beta_j^{(2)})$, then $[\psi]$ is replaced by $(\phi_j^{(2)})^{k_2}[\psi](\phi_j^{(1)})^{k_1}$ for some $k_1,k_2\in\mathbb{Z}$. It is easy to check that

$$\Big((\phi_j^{(2)})^{k_2}[\psi](\phi_j^{(1)})^{k_1}\Big)\phi_j^{(1)}=\phi_j^{(2)}\Big((\phi_j^{(2)})^{k_2}[\psi](\phi_j^{(1)})^{k_1}\Big).$$

Besides, the compatibility does not depend on the choice of global monodromies. Indeed, a different pair of global monodromies replaces $\phi_j^{(1)}$ and $\phi_j^{(2)}$ with $Q_1^{-1}\phi_j^{(1)}Q_1$ and $Q_2^{-1}\phi_j^{(2)}Q_2$, respectively. The set of all possibilities for $[\psi]$ is

$$\left\{\Big(Q_2^{-1}\phi_j^{(2)}Q_2\Big)^{k_2}Q_2^{-1}[\psi]Q_1\Big(Q_1^{-1}\phi_j^{(1)}Q_1\Big)^{k_1}\,\middle|\,k_1,k_2\in\mathbb{Z}\right\}.$$

It is easy to check that

$$\begin{aligned}\Big(\big(Q_2^{-1}\phi_j^{(2)}Q_2\big)^{k_2}Q_2^{-1}[\psi]Q_1\big(Q_1^{-1}\phi_j^{(1)}Q_1\big)^{k_1}\Big)&Q_1^{-1}\phi_j^{(1)}Q_1\\ &=Q_2^{-1}\phi_j^{(2)}Q_2\Big(\big(Q_2^{-1}\phi_j^{(2)}Q_2\big)^{k_2}Q_2^{-1}[\psi]Q_1\big(Q_1^{-1}\phi_j^{(1)}Q_1\big)^{k_1}\Big).\end{aligned}$$

Theorem 2.2.7. *Let $f_1:M_1\to S^2$ and $f_2:M_2\to S^2$ be torus fibrations with branch sets $\mathcal{B}_1=\{p_j^{(1)}\}$ and $\mathcal{B}_2=\{p_j^{(2)}\}$, $|\mathcal{B}_1|=n=|\mathcal{B}_2|$, with global monodromies $(\phi_1^{(1)},\ldots,\phi_n^{(1)})$ and*

$(\phi_1^{(2)}, \ldots, \phi_n^{(2)})$ *such that each singular fibre is locally symmetric. Suppose that, for each j, there exists a locally fibre-preserving homeomorphism between $f_1^{-1}(p_j^{(1)})$ and $f_2^{-1}(p_j^{(2)})$ compatible with the given global monodromies. Let $\widetilde{f}_1 = f_1 \oplus f^L_{\mathcal{O}(f_1)} : \widetilde{M}_1 \to S^2$ and $\widetilde{f}_2 = f_2 \oplus f^L_{\mathcal{O}(f_2)} : \widetilde{M}_2 \to S^2$ be direct sums. Then $(\widetilde{M}_1, \widetilde{f}_1)$ and $(\widetilde{M}_2, \widetilde{f}_2)$ are fibre-preserving homeomorphic.*

Remark 2.2.8. The one-to-one correspondence between singular fibres via locally fibre-preserving homeomorphisms in the hypothesis of Theorem 2.2.7 implies that $\mathcal{O}(f_1) = \mathcal{O}(f_2)$.

The following definition first appeared in Part II, Definition 4 in [Moi77].

Definition 2.2.9. Suppose that $f : M \to S^2$ is a torus fibration with branch set $\mathcal{B} = \{p_j\}$. Let $\alpha : S^1 \to \mathrm{Homeo}_0(\mathbb{T}^2)$ be a closed curve in the group of homeomorphisms of $\mathbb{T}^2$ isotopic to the identity. Let $D \subset S^2 \setminus \mathcal{B}$ be a disc. Identify ∂D with S^1 and $f^{-1}(D)$ with $D \times \mathbb{T}^2$, then α defines a canonical homeomorphism
$$\widetilde{\alpha} : f^{-1}(\partial D) \to \partial(D \times \mathbb{T}^2).$$
Denote $M_{D,\alpha} = \overline{M \setminus f^{-1}(D)} \cup_{\widetilde{\alpha}} (D \times \mathbb{T}^2)$ and let $f_{D,\alpha} : M_{D,\alpha} \to S^2$ be the map which is equal to f on $\overline{M \setminus f^{-1}(D)}$ and equal to the projection $D \times \mathbb{T}^2 \to D$ on $D \times \mathbb{T}^2$. Thus the map $f_{D,\alpha}$ is a torus fibration, called *the α-twisting of M at D*.

Lemma 2.2.10. *Let $f : M \to S^2$ be a torus fibration and $f_{D,\alpha}$ be an α-twisting of f for some $\alpha : S^1 \to \mathrm{Homeo}_0(\mathbb{T}^2)$ and disc $D \subset S^2$. Suppose that f has surjective monodromy homomorphisms. Then f and $f_{D,\alpha}$ are fibre-preserving homeomorphic.*

Proof. See Proposition 2.1 in [Fun22]. □

Proof of Theorem 2.2.7. By Theorem 2.1.12, we suppose that $(\widetilde{M}_1, \widetilde{f}_1)$ and $(\widetilde{M}_2, \widetilde{f}_2)$ have the same branch set $\mathcal{B} = \{p_1, \ldots, p_{l+m}\}$. Taking the base point $p \in S^2 \setminus \mathcal{B}$ and loops $\gamma_1, \ldots, \gamma_{l+m}$ based at p, we further suppose that the global monodromies of $(\widetilde{M}_1, \widetilde{f}_1)$ and $(\widetilde{M}_2, \widetilde{f}_2)$ determined by p, $\gamma_1, \ldots, \gamma_{l+m}$ coincide, say
$$(g_1, \ldots, g_l) \bullet (\phi_1, \ldots, \phi_m).$$

Since the compatibility of a locally fibre-preserving homeomorphism does not depend on the global monodromy, we may assume that there exists the permutation $\sigma \in S_m$ such that

- $\widetilde{f}_1^{-1}(p_j)$ and $\widetilde{f}_2^{-1}(p_j)$ are locally fibre-preserving homeomorphic compatible with the global monodromies, for $j = 1, \ldots, l$;
- $\widetilde{f}_1^{-1}(p_j)$ and $\widetilde{f}_2^{-1}(p_{l+\sigma(j-l)})$ are locally fibre-preserving homeomorphic compatible with the global monodromies, for $j = l+1, \ldots, l+m$.

Let $G = \mathbb{Z}\epsilon_1 + \ldots + \mathbb{Z}\epsilon_{l+m}$ be the free group of rank $l+m$. Consider the tuples
$$\Big((\epsilon_1, g_1), \ldots, (\epsilon_l, g_l), (\epsilon_{l+1}, \phi_1), \ldots, (\epsilon_{l+m}, \phi_m)\Big),$$
$$\Big((\epsilon_1, g_1), \ldots, (\epsilon_l, g_l), (\epsilon_{l+\sigma(1)}, \phi_1), \ldots, (\epsilon_{l+\sigma(m)}, \phi_m)\Big)$$
in $G \times \mathrm{SL}(2, \mathbb{Z})$. The swappability of the global monodromy implies that one tuple can be transformed into the other by elementary transformations. Thus, using a fibre-preserving homeomorphism, we may suppose that σ is the identical permutation.

Consider the locally fibre-preserving homeomorphisms $\Psi_{S,j} : U_j^{(1)} \to U_j^{(2)}$, $\Psi_{M,j} : \widetilde{f}_1^{-1}(U_j^{(1)}) \to \widetilde{f}_2^{-1}(U_j^{(2)})$ with sufficiently small neighbourhoods $U_j^{(i)}$ of p_j, for $j = 1, \ldots, l+m$ and $i = 1, 2$. They extend to locally fibre-preserving homeomorphisms $\Psi'_{S,j} : U_j^{(1)} \cup \beta_j^{(1)} \to U_j^{(2)} \cup \beta_j^{(2)}$, $\Psi'_{M,j} : \widetilde{f}_1^{-1}(U_j^{(1)} \cup \beta_j^{(1)}) \to \widetilde{f}_2^{-1}(U_j^{(2)} \cup \beta_j^{(2)})$ where, for $i = 1, 2$, $\beta_j^{(i)}$ is a path joining p to some point $d_j^{(i)} \in \partial U_j^{(i)}$ such that $\beta_1^{(i)}, \ldots, \beta_{l+m}^{(i)}, U_1^{(i)}, \ldots, U_{l+m}^{(i)}$ are disjoint away from p and $d_1^{(i)}, \ldots, d_{l+m}^{(i)}$.

The mapping class represented by $\varphi_j = \Psi'_{M,j} |_{\widetilde{f}_1^{-1}(p)}$ satisfies $[\varphi_j] g_j = g_j [\varphi_j]$ if $j = 1, \ldots, l$ or $[\varphi_j]\phi_{j-l} = \phi_{j-l}[\varphi_j]$ otherwise. All singular fibres are locally symmetric. Set $\Gamma_i = \bigcup_j U_j^{(i)} \cap \beta_j^{(i)}$ for $i = 1, 2$. Therefore we obtain a fibre-preserving homeomorphism $\Psi_S : \Gamma_1 \to \Gamma_2$, $\Psi_M : \widetilde{f}_1^{-1}(\Gamma_1) \to \widetilde{f}_2^{-1}(\Gamma_2)$. One may further assume that $\Gamma_1 = \Gamma = \Gamma_2$ without loss of generality.

It remains to prove that $\widetilde{f}_1 |_\Gamma$ and $\widetilde{f}_2 |_\Gamma$ extend to the unique torus fibration over the complementary disc of Γ within S^2, up to fibre-preserving homeomorphism, but this follows from Lemma 2.2.10. □

2.2.3 Singular fibrations and singularities

This subsection introduces *singular fibrations* and illustrates Corollary B.

Let $f : M^4 \to S^2$ be a smooth map between a connected closed oriented 4-manifold M^4 and the 2-sphere with finitely many critical points, with generic fibre F^2. Church and Timourian proved that each singularity p of f is *cone-like*, i.e. the singularity p admits a cone neighbourhood in the singular fibre $V = f^{-1}(f(p))$; see [CT74, Lemma 2.1 and (Lemma) 2.4] and also see [Fun11, p.835-836].

Isolated singularities are separated. In fact, there exist arbitrarily small adapted neighbourhoods of cone-like singularities, as introduced by King in [Kin78, p.396]. An *adapted neighbourhood* around a singularity $p \in M^4$ is a compact neighbourhood $Z^4 \subset M^4$ satisfying the following :

1) The restriction $f|_{Z^4} : Z^4 \to D^2$ is a proper map onto a disk $D^2 \subset S^2$;
2) The fibre $f^{-1}(x)$ is transversal to ∂Z^4 for each $x \in int(D^2)$ and $E = f^{-1}(S^1) \cap Z^4 \subset \partial Z^4$;
3) Set $V = f^{-1}(f(p))$ and $K = V \cap \partial Z^4$. Then $N(K) = f^{-1}(D_0^2) \cap \partial Z^4$ is a tubular neighbourhood of K within ∂Z^4 endowed with a trivialization $\theta : N(K) \to K \times D_0^2$ induced by f, where $D_0^2 \subset D^2$ is a sufficiently small disk containing $f(p)$.
4) The composition $f_K = r \circ f : \partial Z^4 \setminus K \to D^2 \to S^1$ is a locally trivial fibration over S^1, where r is the radical projection ;
5) The data $(\partial Z^4, K, f_K, \theta)$ is an open book decomposition.

It is equivalent to the date (f_Z, Φ) satisfying the following :

1) The map $f_Z : Z^4 \to D^2$ is proper and induced by f. Set $V = f_Z^{-1}(f_Z(p))$, $K = V \cap \partial Z^4$ and $E^3 = f_Z^{-1}(S^1) \subset \partial Z^4$. Then the restriction $f_Z : E^3 \to S^1$ is a fibration with fibre F_p^2.
2) The flow Φ on Z^4 is continuous along directions parallel to D^2 such that
   i) $f(\Phi(z, d)) = f(z) + d$ for $z \in Z^4$ and $d \in D^2$ when both sides are within Z^4 ;
   ii) the mapping $(x, t) \mapsto \Phi(x, -tf_Z(x))$ is a homeomorphism from $E^3 \times [0, 1)$ to $Z^4 \setminus V$;
   iii) there exists a vanishing compact subset $\mathcal{A} \subset E^3$ such that $x \mapsto \Phi(x, -f_Z(x))$ induces a homeomorphism from $E^3 \setminus \mathcal{A}$ to $V \setminus p$ and sends $\mathcal{A}$ to p.

King proved that, for the fibration of a manifold M^m in dimension $m \neq 4, 5$, one can always find adapted neighbourhoods for singularities diffeomorphic to the m-disk. In dimension 4, however, adapted neighbourhoods can only be supposed to be contractible. We call a singularity *regular* if it admits an arbitrarily small adapted neighbourhood which is diffeomorphic to the 4-disk.

Definition 2.2.11. A smooth map $f : M^4 \to S^2$ between a connected closed oriented 4-manifold and the 2-sphere is a *singular fibration* if it has only finitely many critical points, all of them being regular.

The binding $K \subset \partial Z^4$ of an open book decomposition is a fibered link. Each fibre of f_K is a surface that has the boundary K and is homotopic to the local Milnor fibre F_p^2. It is proved in [Kin78, Theorem 1] that the local mapping torus E^3 and the vanishing compact subset $\mathcal{A} \subset E^3$ up to isotopy form a complete invariant of the adapted neighbourhood up to fibre-preserving homeomorphism. In particular, if a singular fibre contains only one singularity and the fibre monodromy is given, then the singular fibre is determined by the isotopy class of local Milnor fibre, up to fibre-preserving homeomorphism.

In general, there could be many singularities in a singular fibre, say $p_1, \ldots, p_n$. The horizontal homeomorphisms given by disjoint adapted neighbourhoods reveal that the local Milnor fibres $F_{p_1}^2, \ldots, F_{p_n}^2$ are disjoint compact subsurfaces embedded in the generic fibre F^2 of the fibration. The fibre monodromy around the singular fibre is a mapping class of the generic fibre F^2, denoted by $\phi_{f^{-1}(f(p_i))}$. The inclusions $\iota_i : F_{p_i}^2 \hookrightarrow F$ induce the homomorphisms $\mathrm{Mod}(F_{p_i}^2) \to \mathrm{Mod}(F^2)$ which send the local monodromies $\phi_{F_{p_i}^2}$ of the mapping tori $E^3 \to S^1$ to mapping classes of the generic fibre. Therefore

$$\phi_{f^{-1}(f(p_i))} = \iota_{1,*}(\phi_{F_{p_1}^2}) \circ \ldots \circ \iota_{n,*}(\phi_{F_{p_n}^2}), \tag{2.2}$$

which does not depend on the order. Furthermore, the following should be well-known.

Lemma 2.2.12. *In a singular fibration $f : M^4 \to S^2$, each local Milnor fibre of an adapted neighbourhood diffeomorphic to the 4-disk is connected with a non-empty boundary.*

Proof. Suppose that p is a singularity of the singular fibration $f : M^4 \to S^2$. If we assume that the binding link K of a singularity is vacuous, the adapted neighbourhood implies a locally trivial fibre bundle $S^3 \to S^1$, which is a contradiction. Since the completion of the local Milnor fibre has a non-empty boundary, the reduced cohomology group $\tilde{H}^2(\overline{F_p^2})$ is trivial. We use Alexander duality and obtain that $\tilde{H}_0(S^3 \setminus \overline{F_p^2})$ is trivial. Hence F_p^2 is connected. □

Definition 2.2.13. A continuous map $g_1 : X_1 \to Y_1$ is *locally topologically equivalent* at $x_1 \in X_1$ to a continuous map $g_2 : X_2 \to Y_2$ at $x_2 \in X_2$ if there exist sufficiently small open neighbourhoods U_1 of x_1, U_2 of x_2, V_1 of $g_1(x_1)$, V_2 of $g_2(x_2)$ and homeomorphisms $\alpha : U_1 \to U_2$, $\beta : V_1 \to V_2$ such that $\beta \circ g_1 |_{U_1} = g_2 \circ \alpha |_{U_1}$.

A point at which f fails to be locally topologically equivalent to the projection $\mathbb{R}^4 \to \mathbb{R}^2$ is called a *branch point*, which is necessarily a singularity. Church and Lamotke have shown that a local Milnor fibre is diffeomorphic to the 2-disk if and only if the associated singularity is not a branch point ; see [CL75, Proposition p.151]. We conclude that, up to fibre-preserving homeomorphism, one may assume that a torus singular fibration has no local Milnor fibre of genus 0 with only 1 boundary component.

Local Milnor fibre of genus zero

When the local Milnor fibre F^2_p is a genus zero surface with $r \geq 2$ boundary components, then $E^3 \cong \partial Z^4 \setminus K$ is the mapping torus of some mapping class $\phi_{F^2_p}$ that is identical on boundary, denoted by $\mathcal{M}_{\phi_{F^2_p}}$. The group of mapping classes identical on the boundary, denoted by $\mathrm{Mod}^*(F^2_p)$, is generated by Dehn twists along the following loops ; see [Waj99].

- Loops $\delta_{i,j}$, $2 \leq i < j \leq r$, that each separates two boundary components from the others.
- Peripheral loops $\alpha_2, \ldots, \alpha_r$, that are parallel to the latter $r-1$ boundary components.

The peripheral loops are mutually disjoint and they keep away from the loops $\delta_{i,j}$. Therefore, $\phi_{F^2_p}$ is the composition of the product of (positive and negative) Dehn twists along peripheral loops and a mapping class $\varphi_{F^2_p}$ generated by the Dehn twists along the rest loops, denoted by

$$\phi_{F^2_p} = (\prod_{i=2}^{r} T^{u_i}_{\alpha_i})\varphi_{F^2_p}$$

with $u_2, \ldots, u_r \in \mathbb{Z}$. The following shows a necessary property for local Milnor fibres of genus zero in a fibration $f : M^4 \to S^2$.

Lemma 2.2.14. *Let $f : M^4 \to S^2$ be a smooth map between a connected closed oriented 4-manifold M^4 and the 2-sphere. Let $p \in M^4$ be an isolated singularity. Given a contractible adapted neighbourhood of p, if the local Milnor fibre F^2_p is a genus zero surface with $r \geq 2$ boundary components and the local monodromy is given by $\phi_{F^2_p} = \prod_{i=2}^{r} T^{u_i}_{\alpha_i}$ with $u_2, \ldots, u_r \in \mathbb{Z}$, then $u_i = \pm 1$, $i = 2, \ldots, r$.*

Proof. The first homology group of F^2_p is isomorphic to $\mathbb{Z}^{r-1}$ and generated by the cycles around boundary components, but excluding the first component. Therefore, $\phi_{F^2_p,*} = id_{H_1(F^2_p,\mathbb{Z})}$ and the homology group $H_1(\mathcal{M}_{\phi_{F^2_p}}, \mathbb{Z}) = H_1(F^2_p, \mathbb{Z}) \rtimes_{\phi_{F^2_p},*} \langle [\gamma] \rangle$ is isomorphic to $\mathbb{Z}^r$, where γ is the closed curve in the mapping torus induced by a fixed point on the first boundary component of F^2_p.

We write $H_1(\mathcal{M}_{\phi_{F^2_p}}, \mathbb{Z}) = \langle a_2, \ldots, a_r, t \rangle$. The boundary ∂Z^4 of the adapted neighbourhood is the union of the mapping torus $\mathcal{M}_{\phi_{F^2_p}}$ and r more solid tori, which is a homology 3-sphere. The inclusion mapping the connected components of the intersection to the mapping torus derives from (positive or negative) powers of the Dehn twist along peripheral loops, which are denoted by $T^{u_1}_{\alpha_1}, \ldots, T^{u_r}_{\alpha_r}$ respectively. By Mayer-Vietoris we have

$$H_2(\partial Z^4, \mathbb{Z}) \longrightarrow H_1(\mathbb{T}^2, \mathbb{Z})^r \xrightarrow{\tau} H_1(\mathcal{M}_{\phi_{F^2_p}}, \mathbb{Z}) \oplus H_1(S^1 \times D^1)^r \longrightarrow H_1(\partial Z^4, \mathbb{Z})$$

where τ is an isomorphism. After the choice of the natural basis, the corresponding $(2r) \times (2r)$-

matrix is given by

$$A = \begin{bmatrix} -1 & 0 & 1 & u_2 & & & & & & & \\ -1 & 0 & & & 1 & u_3 & & & & & \\ -1 & 0 & & & & & 1 & u_4 & & & \\ \vdots & & & & & & & & \ddots & & \\ -1 & 0 & & & & & & & \cdots & 1 & u_r \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & 0 & & & & & & & & & \\ & & 1 & 0 & & & & & & & \\ & & & & 1 & 0 & & & & & \\ & & & & & & 1 & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 1 & 0 \end{bmatrix}$$

satisfying $det(A) = \pm 1$. It follows that $u_2 \cdots u_r = \pm 1$. □

In particular, we have the following consequence.

Corollary 2.2.15. *Let $f : M^4 \to S^2$ be a smooth map between a connected closed oriented 4-manifold and the 2-sphere. Given a contractible adapted neighbourhood of a singularity, if the local Milnor fibre is a genus zero surface with exactly two boundary components, the local monodromy is either a positive or a negative Dehn twist.*

Proof. In this case, the local Milnor fibre F_p^2 is an annulus whose mapping class group is generated by the Dehn twist along the unique peripheral loop. By Lemma 2.2.14 we have $u_2 = \pm 1$. Hence $\phi_{F_p^2}$ is either the positive or the negative Dehn twist. □

Local Milnor fibre of genus one

We consider the case when the local Milnor fibre for a contractible adapted neighbourhood of a cone-like singularity in $f : M^4 \to S^2$ is a torus with $r \geq 1$ disks removed, say $F_p^2 = \mathbb{T}^2 \setminus (D_1 \sqcup \ldots \sqcup D_r)$. Again, let $\phi_{F_p^2} \in \mathrm{Mod}(F_p^2)$ be the local monodromy. By the Mayer–Vietoris sequence on ∂Z^4 we have

$$H_2(\partial Z^4, \mathbb{Z}) \longrightarrow H_1(\mathbb{T}^2, \mathbb{Z})^r \longrightarrow H_1(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z}) \oplus H_1(S^1 \times D^1, \mathbb{Z})^r \longrightarrow H_1(\partial Z^4, \mathbb{Z}).$$

Since the boundary ∂Z^4 is a homology 3-sphere,$H_1(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^r$.

Now we compute the homology group $H_1(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z})$ of the mapping torus. Write $\mathcal{M}_{\phi_{F_p^2}}$ as the union of $A = F_p^2 \times I_1$ and $B = F_p^2 \times I_2$ and take the inclusion maps $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow \mathcal{M}_{\phi_{F_p^2}}$ and $l : B \hookrightarrow \mathcal{M}_{\phi_{F_p^2}}$. By Mayer–Vietoris we have

$$\begin{aligned} \longrightarrow & H_1(A \cap B, \mathbb{Z}) \xrightarrow{(i_*, j_*)} H_1(A, \mathbb{Z}) \oplus H_1(B, \mathbb{Z}) \xrightarrow{k_* - l_*} H_1(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z}) \xrightarrow{\partial_*} \\ & H_0(A \cap B, \mathbb{Z}) \xrightarrow{(i_*, j_*)} H_0(A, \mathbb{Z}) \oplus H_0(B, \mathbb{Z}) \xrightarrow{k_* - l_*} H_0(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z}) \longrightarrow 0. \end{aligned}$$

Notice that $im\partial_*$ is isomorphic to $ker(H_0(A \cap B, \mathbb{Z}) \xrightarrow{(i_*, j_*)} H_0(A, \mathbb{Z}) \oplus H_0(B, \mathbb{Z})) \simeq \mathbb{Z}$. To ensure that $H_1(\mathcal{M}_{\phi_{F_p^2}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^r$, we require that $ker\partial_* \simeq \mathbb{Z}^{r-1}$ and therefore

$$im(H_1(A \cap B, \mathbb{Z}) \xrightarrow{(i_*, j_*)} H_1(A, \mathbb{Z}) \oplus H_1(B, \mathbb{Z})) \simeq \mathbb{Z}^{3+r}.$$

Lemma 2.2.16. *Let $f : M^4 \to S^2$ be a smooth map between a connected closed oriented 4-manifold M^4 and the 2-sphere. Let p be a singularity of f with a contractible adapted neighbourhood. Suppose that the local Milnor fibre F_p^2 is a torus with a disk removed and consider the inclusion $\iota : F_p^2 \hookrightarrow \mathbb{T}^2$. Then the binding link $K \subset \partial Z^4 \cong S^3$ is either the trefoil knot or the figure-eight knot. Furthermore, the local monodromy $\phi_{F_p^2}$ induces a mapping class of the torus $\iota_*(\phi_{F_p^2}) \in \mathrm{Mod}(\mathbb{T}^2) \simeq \mathrm{SL}(2, \mathbb{Z})$ which is conjugate to one of*

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Proof. We only prove the assertion of the local monodromy and a complete proof has been introduced by Burde and Zieschang (see Proposition 5.14 in [BZ03]).

The mapping class group of F_p^2 is generated by the Dehn twists along two intersecting loops α, β and the Dehn twist along the peripheral loop δ. With a careful arrangement, the peripheral loop is away from the others and therefore the local monodromy is the composition $\phi_{F_p^2} = T_\delta^u \circ \varphi_{F_p^2}$ with $u \in \mathbb{Z}$ and $\varphi_{F_p^2}$ generated by the Dehn twists along α, β. Thus, along the inclusion $\iota : F_p^2 \hookrightarrow \mathbb{T}^2$, the pushforward $\iota_*(\phi_{F_p^2})$ is equal to the pushforward $\iota_*(\varphi_{F_p^2})$. Fix the isomorphism between $\mathrm{Mod}(\mathbb{T}^2)$ and $\mathrm{SL}(2,\mathbb{Z})$ such that the induced homomorphism $\mathrm{Mod}(F_p^2) \to \mathrm{SL}(2,\mathbb{Z})$ sends T_α (resp. T_β) to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ (resp. } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\text{)}.$$

Suppose that $\iota_*(\phi_{F_p^2}) \in \mathrm{Mod}(\mathbb{T}^2)$ is expressed by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

We take the basis of the homology group $H_1(F_p^2)$ consisting of the cycles which are parallel with α and β, which further determines the bases of $H_1(A,\mathbb{Z})$, $H_1(B,\mathbb{Z})$ and $H_1(A \cap B,\mathbb{Z})$. The pushforward $\phi_{F_p^2,*} : H_1(F_p^2,\mathbb{Z}) \to H_1(F_p^2,\mathbb{Z})$ is again expressed by A. The homomorphism $H_1(A \cap B,\mathbb{Z}) \xrightarrow{(i_*,j_*)} H_1(A,\mathbb{Z}) \oplus H_1(B,\mathbb{Z})$ is an isomorphism whose corresponding 4×4-matrix is given by

$$\begin{bmatrix} I & A \\ I & I \end{bmatrix}.$$

satisfying $det(I - A) = \pm 1$. Hence $a + d = 1$ or 3. □

Conversely, we do have a connected closed oriented 4-manifold M^4 with a singular fibration $f : M^4 \to S^2$ that has the singularities as desired. Both the trefoil knot and the figure-eight knot are defined by the links of polynomial maps $\mathbb{R}^4 \to \mathbb{R}^2$ with an isolated critical point at 0. An explicit realisation of the trefoil knot was first given by Brauner (see [Bra28]) who constructs a complex polynomial

$$(f_{\text{Brauner}} : \mathbb{C}^2 \to \mathbb{C}) : (u, v) \mapsto u^2 - v^3.$$

Perron found the first realisation of the figure-eight knot in [Per82].

We end with the proof of Corollary B.

Proof of Corollary B. Without loss of generality, we assume that there does not exist any local Milnor fibre of genus 0 with only 1 boundary components. By Corollary 2.2.15 and Lemma 2.2.16, the type of singularities $\mathcal{O}(f_1) = \mathcal{O}(f_2)$ consists of simple conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$.

All singular fibres are locally symmetric. If the local Milnor fibre is an annulus, then a mapping class that commutes with the fibre monodromy preserves this annulus up to isotopy. If the local Milnor fibre is a torus with a disc removed, then no mapping class changes the local Milnor fibre up to isotopy.

Any pair of singular fibres with conjugate fibre monodromies has local Milnor fibres compatible with their fibre monodromies, so they have the same local Milnor fibre up to isotopy. Therefore, there exists a local fibre-preserving homeomorphism compatible with their fibre monodromies.

The corollary follows from Theorem C and Theorem 2.2.7. □

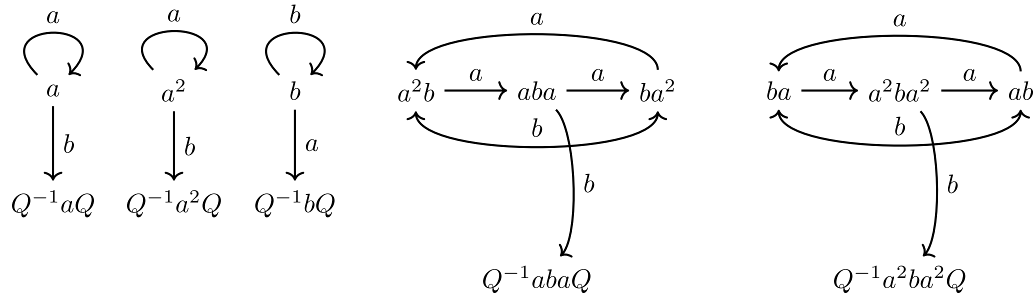
2.3 Theorem of R. Livné, complement and extension

In this section, G is the modular group $\mathrm{PSL}(2,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, which we represent as $\langle a, b \mid a^3 = b^2 = 1 \rangle$. Each element in G has the unique *reduced form* as a word in $\{a, a^2, b\}$ where b's and powers of a appear alternatively. The *length* of an element $g \in G$ is defined as the length of its reduced form, denoted by $l(g)$.

Recall that elements in

$$\mathcal{S} = \{a, a^2, b, s_0 = a^2b, s_1 = aba, s_2 = ba^2, t_0 = ba, t_1 = a^2ba^2, t_2 = ab\} \subset G$$

are "*short*"; the rest conjugates of short elements in G are called "*long*", which are expressed by $Q^{-1}a^\epsilon Q$, $Q^{-1}bQ$ or $Q^{-1}a^\epsilon ba^\epsilon Q$ with $\epsilon = 1, 2$ and $l(Q) \geq 1$. The following diagram shows all conjugates of short elements and their conjugates with a, a^2 and b.



Note that there are nice circuits along s_0, s_1, s_2 and t_0, t_1, t_2. In fact, we will see a lot of symmetric properties on them. For convenience, the subscripts are regarded as elements in $\mathbb{Z}/3\mathbb{Z}$ and represented by $0, 1, 2$ without further explanations.

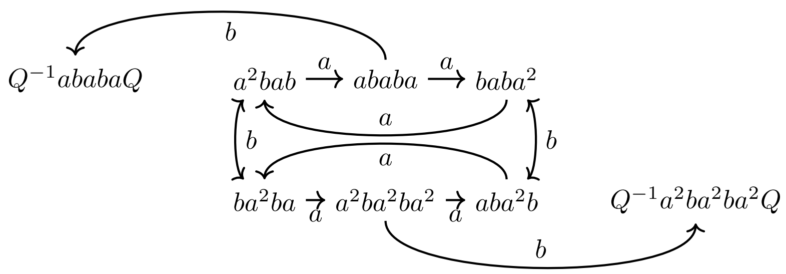
Recall that elements in

$$\mathcal{S}_2 = \mathcal{S} \cup \{bab, ba^2b, a^2ba, aba^2, a^2bab, ababa, baba^2, ba^2ba, a^2ba^2ba^2, aba^2b\}$$

are "almost short"; the rest conjugates of almost short elements in G are called "almost long", which are expressed by

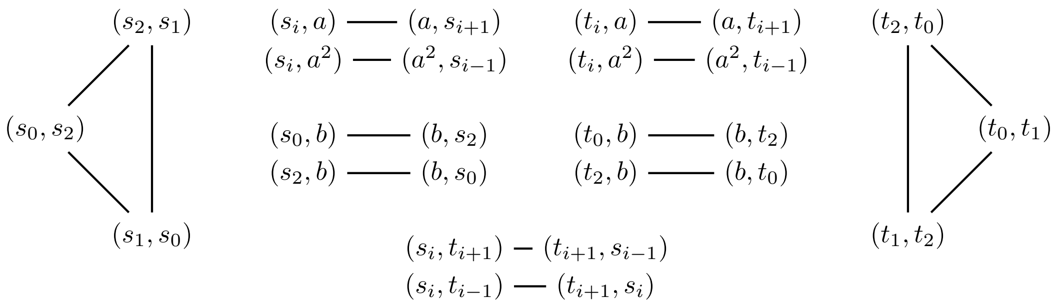
$$Q^{-1}ba^\epsilon bQ, Q^{-1}a^\epsilon ba^\epsilon Q, Q^{-1}a^\epsilon ba^{-\epsilon}Q \text{ or } Q^{-1}a^\epsilon ba^\epsilon ba^\epsilon Q$$

with $\epsilon = 1, 2$ and $l(Q) \geq 1$. The almost short elements correspond to six conjugacy classes of G, five of which have been illustrated and the following is the last one.



Recall that elementary transformations R_i, $1 \leq i \leq n-1$ on n-tuples in G send $(g_1, \dots g_n)$ to $(g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1}g_ig_{i+1}, g_{i+2}, \dots, g_n)$ respectively. The inverse of R_i is given by R_i^{-1} sending $(g_1, \dots, g_n)$ to $(g_1, \dots, g_{i-1}, g_ig_{i+1}g_i^{-1}, g_i, g_{i+2}, \dots, g_n)$. Both R_i and R_i^{-1} are called elementary transformations. Especially, we will neither apply R_i if $g_i = 1$ nor apply R_i^{-1} if $g_{i+1} = 1$, but use R_i^{-1} and R_i instead respectively to avoid troubles.

Elementary transformations introduce many elegant substitutions for pairs of short elements. Here we list some substitutions in the following graphs for readers unfamiliar with them.



Definition 2.3.1. An n-tuple $(g_1, \dots, g_n)$ in G is said to be *inverse-free* if, applying any finite sequence of elementary transformations, the resulting n-tuple satisfies the following requirements :

- it contains no adjacent elements which are mutually inverse;
- it contains no sub-triple of the form (h, h, h) with $h^3 = 1$.

For instance, (s_1, t_1), (a, a^2), (b, b), (a, a, a) and their concatenations are not inverse-free.

Theorem 2.3.2 (Livné)**.** *Let $g_1, \dots, g_n$ be conjugates of s_1 such that $g_1 \cdots g_n = 1$. Then, the n-tuple $(g_1, \dots, g_n)$ is Hurwitz equivalent to an n-tuple $(h_1, \dots, h_n)$ with each h_i short (i.e. the component h_i is equal to one of s_0, s_1 and s_2).*

Moishezon showed a proof of Theorem 2.3.2 and introduced the following complement in [Moi77].

Theorem 2.3.3 (Moishezon). *Let $h_1, \dots, h_n$ be such that each of h_i, $i = 1, \dots, n$, is equal to one of s_0, s_1 and s_2 satisfying $h_1 \cdots h_n = 1$. Then, $n \equiv 0 \pmod 6$ and the n-tuple $(h_1, \dots, h_n)$ is Hurwitz equivalent to $(s_0, s_2)^{n/2}$.*

In this section, We first extend the above theorems for $(g_1, \dots, g_n)$ with each g_i conjugate to some short element, then we show a similar result for $(g_1, \dots, g_n)$ when each g_i is conjugate to some almost short element.

2.3.1 Tuples of short elements

Recall the set of short elements is $\mathcal{S} = \{a, a^2, b, s_0, s_1, s_2, t_0, t_1, t_2\}$. We first show that an inverse-free tuple of short elements cannot contain both s_i and t_j for any $(i, j) \in (\mathbb{Z}/3\mathbb{Z})^2$.

Proposition 2.3.4. *Let $g_1, \dots, g_n$ be short satisfying at most one of them is equal to one of a, a^2, b and $g_1 \cdots g_n = 1$. Suppose that $(g_1, \dots, g_n)$ is inverse-free. Then, either each of g_i, $i = 1, \dots, n$ is equal to one of a, a^2, b, s_0, s_1, s_2 or each of g_i, $i = 1, \dots, n$ is equal to one of a, a^2, b, t_0, t_1, t_2.*

Proof. Assume that at least one of $g_1, \dots, g_n$ is conjugate to s_0 and at least one of $g_1, \dots, g_n$ is conjugate to t_0. The substitution of (s_k, t_{k+1}), (t_{k+1}, s_{k-1}) and the substitution of (s_k, t_{k-1}), (t_{k+1}, s_k) imply that $(g_1, \dots, g_n)$ can be transformed by elementary transformations into $(h_1, \dots, h_n)$ with $p, q \geq 1$, $p+q \in \{n-1, n\}$ such that $h_1 \in \{a, a^2, b, s_0, s_1, s_2\}$, $h_i \in \{s_0, s_1, s_2\}$ for $i = 2, \dots, n-q$ and $h_i \in \{t_0, t_1, t_2\}$ for $i = n-q+1, \dots, n$. Let $\mathcal{A}$ be the set of elements in $\{h_1, \dots, h_n\}$, $\mathcal{A}_s = \mathcal{A} \cap \{s_0, s_1, s_2\}$ and $\mathcal{A}_t = \mathcal{A} \cap \{t_0, t_1, t_2\}$. The inverse-freeness requires that s_j and t_j cannot appear together in $\mathcal{A}$.

Assume that $|\mathcal{A}_s| = 1 = |\mathcal{A}_t|$. Then the product $h_1 \cdots h_n$ is expressed by $\tilde{h} s_j^u t_k^v$ with $\tilde{h} \in \{1, a, a^2, b\}$, $u, v \geq 1$ and $j \neq k$. To ensure that $(g_1, \dots, g_n)$ is inverse-free, the product must be one of the following forms with $u, v \geq 1$.

$$\begin{aligned}
\tilde{h} = 1 \quad &\Rightarrow \quad s_0^u t_1^v = (a^2 b)^u (a^2 b a^2)^v, \\
&\quad\;\;\; s_0^u t_2^v = (a^2 b)^u (ab)^v, \\
&\quad\;\;\; s_1^u t_0^v = (aba)^u (ba)^v, \\
&\quad\;\;\; s_1^u t_2^v = (aba)^u (ab)^v, \\
&\quad\;\;\; s_2^u t_0^v = (ba^2)^u (ba)^v, \\
&\quad\;\;\; s_2^u t_1^v = (ba^2)^u (a^2 b a^2)^v. \\
\tilde{h} = a \quad &\Rightarrow \quad a s_0^u t_1^v = a(a^2 b)^u (a^2 b a^2)^v = b(a^2 b)^{u-1} (a^2 b a^2)^v, \\
&\quad\;\;\; a s_1^u t_2^v = a(aba)^u (ab)^v, \\
&\quad\;\;\; a s_2^u t_0^v = a(baa)^u (ba)^v. \\
\tilde{h} = a^2 \quad &\Rightarrow \quad a^2 s_0^u t_2^v = a^2 (a^2 b)^u (ab)^v, \\
&\quad\;\;\; a^2 s_1^u t_0^v = a^2 (aba)^u (ba)^v = ba(aba)^{u-1} (ba)^v, \\
&\quad\;\;\; a^2 s_2^u t_1^v = a^2 (ba^2)^u (a^2 b a^2)^v. \\
\tilde{h} = b \quad &\Rightarrow \quad b s_0^u t_1^v = b(a^2 b)^u (a^2 b a^2 b)^v, \\
&\quad\;\;\; b s_1^u t_0^v = b(aba)^u (ba)^v, \\
&\quad\;\;\; b s_1^u t_2^v = b(aba)^u (ab)^v, \\
&\quad\;\;\; b s_2^u t_1^v = b(ba^2)^u (a^2 b a^2)^v = a^2 (ba^2)^{u-1} (a^2 b a^2)^v.
\end{aligned}$$

However, each of them cannot express 1, which contradicts the fact that $h_1 \cdots h_n = g_1 \cdots g_n = 1$.

Assume that $|\mathcal{A}_s| = 1$ and $|\mathcal{A}_t| = 2$. Pairs of the form (t_j, t_{j+1}) never appear since the substitutions of (t_0, t_1), (t_1, t_2) and (t_2, t_0) imply a contradiction with the inverse-freeness. If $\tilde{h} \neq 1$, then the tuple $(h_1, \dots, h_n)$ is expressed by $(\tilde{h}) \bullet (s_j)^u \bullet (t_{j-1})^v \bullet (t_{j+1})^w$ with $\tilde{h} \in \{a, a^2, b\}$, $u, v, w \geq 1$ and some j. The substitution of (a, s_k), (s_{k-1}, a) and the substitution of (a^2, s_k), (s_{k+1}, a^2) reveal that $\tilde{h} = b$. However, the substitution of (b, s_0), (s_2, b), the substitution of (b, s_2), (s_0, b) and the following substitutions

$$(b) \bullet (s_1)^u \bullet (t_0)^v \bullet (t_2)^w \longrightarrow (b) \bullet (t_0)^w \bullet (s_1)^u \bullet (t_0)^v \longrightarrow (b) \bullet (t_2)^v \bullet (t_0)^w \bullet (s_1)^u$$

further conclude that $\tilde{h} = 1$. Thus, $g_1 \cdots g_n = h_1 \cdots h_n$ is expressed by $s_0^u t_2^v t_1^w$, $s_1^u t_0^v t_2^w$ or $s_2^u t_1^v t_0^w$ each of which cannot express 1, which is a contradiction.

We have a similar argument for the case where $|\mathcal{A}_s| = 2$ and $|\mathcal{A}_t| = 1$. Hence either none of $g_1, \dots, g_n$ is conjugate to s_0 or none of $g_1, \dots, g_n$ is conjugate to t_0. We finish the prove of the proposition. $\square$

As an immediate consequence, we have Lemma 2.3.5.

Lemma 2.3.5. *Let $g_1, \dots, g_n$ be equal to s_0, s_1, s_2, t_0, t_1 or t_2 satisfying $g_1 \cdots g_n = 1$. Suppose that $(g_1, \dots, g_n)$ is inverse-free. Then, $n \equiv 0 \pmod 6$ and the n-tuple $(g_1, \dots, g_n)$ is Hurwitz equivalent to either $(s_0, s_2)^{n/2}$ or $(t_0, t_2)^{n/2}$.*

Proof. By Proposition 2.3.4, either each g_i is equal to one of s_0, s_1, s_2, or each g_i is equal to one of t_0, t_1, t_2. By Theorem 2.3.3, the n-tuple can be transformed by elementary transformations into either $(s_0, s_2, s_0, s_2, s_0, s_2)^{n/6}$ or $(t_0, t_2, t_0, t_2, t_0, t_2)^{n/6}$. $\square$

Note that $s_0 s_2 s_0 s_2 s_0 s_2 = t_0 t_2 t_0 t_2 t_0 t_2 = 1$ and in fact sextuples with alternative s_i's and s_j's (resp. t_i's and t_j's) can be transformed into each other by elementary transformations. In general, we will show the reduced form of the product for a tuple with alternative powers of s_0 and s_2 (resp. powers of t_0 and t_2). We will only prove Proposition 2.3.6 but omit the proof of Proposition 2.3.7, which is quite similar. The idea comes from Moishezon (see [Moi77, p.181-187]) but with a slight modification and a more subtle analysis.

Proposition 2.3.6. *Let $(g_1, \dots, g_n)$ be a tuple of s_0, s_2 with $n \geq 1$ and take $\mu, \nu \geq 1$. Let $\mathcal{T}$ be the set of tuples of s_0, s_2 obtained from $(g_1, \dots, g_n)$ by elementary transformations. Suppose that each tuple in $\mathcal{T}$ satisfies the following requirements :*
i) it starts with at least μ s_2 ;
ii) it ends with at least ν s_0 ;
iii) it contains no consecutive sub-tuples of the form $(s_0, s_2)^3$.
Then, the reduced form of $g_1 \cdots g_n$ is given by $(ba^2)^{\mu-1} bRb(a^2b)^{\nu-1}$ with some $R \in G$.

Proposition 2.3.7. *Let $(g_1, \dots, g_n)$ be a tuple of t_0, t_2 with $n \geq 1$ and take $\mu, \nu \geq 1$. Let $\mathcal{T}$ be the set of tuples of t_0, t_2 obtained from $(g_1, \dots, g_n)$ by elementary transformations. Suppose that each tuple in $\mathcal{T}$ satisfies the following requirements :*
i) it starts with at least μ t_0 ;
ii) it ends with at least ν t_2 ;
iii) it contains no consecutive sub-tuples of the form $(t_0, t_2)^3$.
Then, the reduced form of $g_1 \cdots g_n$ is given by $(ba)^{\mu-1} bRb(ab)^{\nu-1}$ with some $R \in G$.

Proof of Proposition 2.3.6. Using elementary transformations on $(g_1, \dots, g_n)$ we can get different resulting tuples in $\{s_0, s_2\}$, which form the set $\mathcal{T}$. Suppose that $(h_1, \dots, h_n)$ is the maximal among them according to the lexicographical order given by $s_0 < s_2$. We write $(h_1, \dots, h_n)$ in the following form

$$(h_1, \dots, h_n) = \prod_{i=1}^{N} (s_2)^{u_i} \bullet (s_0)^{v_i}$$

with $\sum_{i=1}^{N} (u_i + v_i) = n$, where $u_1 \geq \mu$, $v_N \geq \nu$ and $u_i > 0, v_i > 0$ for all $i = 1, \dots, N$.

Claim 1 : $u_i \geq 2$, $i = 2, 3, \dots, N$.

Claim 2 : $v_1 \geq 2$.

Claim 3 : For $i \in \{1, 2, \dots, N-2\}$, if $v_i = 1$ then $v_{i+1} > 1$.

Claim 4 : For $i \in \{2, 3, \dots, N-1\}$, if $v_i = 1$ then $u_i \geq 3$ and $u_{i+1} \geq 3$.

Claim 1 relies on the maximality of $(h_1, \dots, h_n)$. Claim 2 uses the first hypothesis. The third hypothesis guarantees both Claim 3 and 4. Now, set $Y_i = (s_0)^{v_i} \bullet (s_2)^{u_{i+1}}$ for $i = 1, \dots, N-1$, say it to be of the *second type* if $v_i \geq 2$, the *first type* if $v_i = 1$. Claim 2 shows that Y_1 is of the second type and Claim 3 reveals that there is no adjacent pair in the first type. Hence, we are able to find sub-tuples $Z_1, \dots, Z_M$ of $(h_1, \dots, h_n)$ such that each Z_j, $j = 1, \dots, M$, is either

- equal to some Y_i of the second type with $i \in \{1, \dots, N-1\}$, or
- the concatenation $Y_i \bullet Y_{i+1}$ with $i \in \{1, \dots, N-2\}$ where Y_{i+1} is of the first type

and we can write $(h_1, \dots, h_n)$ in the form

$$(h_1, \dots, h_n) = (s_2)^{u_1} \bullet \prod_{j=1}^{M} Z_j \bullet s_0^{v_N}.$$

For $j = 1, \ldots, M$, if Z_j is equal to some Y_i of the second type, then the product of components of Z_j has the reduced form $a^2 R_j a^2$ with some $R_j \in G$. Indeed, $Z_j = (s_0)^{v_i} \bullet (s_2)^{u_{i+1}}$ with $v_i \geq 2$ and $u_{i+1} \geq 2$, the product of whose components is equal to $(a^2b)^{v_i-1}a(ba^2)^{u_{i+1}-1}$. If $Z_j = Y_i \bullet Y_{i+1}$, $i \in \{1, \ldots, N-2\}$, then the product of its components is given by

$$s_0^{v_i} s_2^{u_{i+1}} s_0^{v_{i+1}} s_2^{u_{i+2}} = (a^2b)^{v_i-1}a(ba^2)^{u_{i+1}-2}a^2(ba^2)^{u_{i+2}-2}$$

with $v_{i+1} = 1$, $u_{i+1} \geq 3$, $u_{i+2} \geq 3$ and $v_i \geq 2$, which also has the reduced form $a^2 R_j a^2$ with some $R_j \in G$. Hence, $g_1 \cdots g_n = h_1 \cdots h_n$ has the form

$$(ba^2)^{u_1} \prod_{j=1}^{M} (a^2 R_j a^2)(a^2 b)^{v_N}$$

with $R_j \in G$, $j = 1, \ldots, M$ where each of $a^2 R_j a^2$ is reduced. □

2.3.2 Conjugates of short elements and tuples

Suppose that $(g_1, \ldots, g_n)$ is an n-tuple with each g_i conjugate to some short element (i.e. the component g_i is conjugate to a, a^2, b or s_1, t_1). In this subsection we show that, in the vast majority of cases, by successive application of elementary transformations the n-tuple can be transformed into an n-tuple of short elements.

Lemma 2.3.8. *Let g_1, g_2, h, $Q' \in G$ be such that $h = g_1 g_2$. Then both*

$$(Q'^{-1}h^{-1}g_1hQ', Q'^{-1}h^{-1}g_2hQ') \text{ and } (Q'^{-1}hg_1h^{-1}Q', Q'^{-1}hg_2h^{-1}Q')$$

are Hurwitz equivalent to $(Q'^{-1}g_1Q', Q'^{-1}g_2Q')$.

Proof. Using R_1^{-2}, we transform

$$(Q'^{-1}h^{-1}g_1hQ', Q'^{-1}h^{-1}g_2hQ')$$

into $(Q'^{-1}g_1Q', Q'^{-1}g_1^{-1}hQ')$. The result is equal to $(Q'^{-1}g_1Q', Q'^{-1}g_2Q')$ as $g_1^{-1}h = g_2$. Similarly

$$(Q'^{-1}hg_1h^{-1}Q', Q'^{-1}hg_2h^{-1}Q')$$

can be transformed into $(Q'^{-1}g_1Q', Q'^{-1}g_2Q')$ by applying R_1^2. □

Lemma 2.3.9. *Let $\epsilon = \pm 1$ and suppose that (τ_1, τ_2) is equal to one of*

$$\{(a^\epsilon ba^\epsilon, a^{-\epsilon}), (a^{-\epsilon}, a^\epsilon ba^\epsilon), (ba^{-\epsilon}, a^{-\epsilon}), (a^{-\epsilon}, a^{-\epsilon}b), (a^\epsilon, ba^\epsilon), (a^\epsilon b, a^\epsilon)\}.$$

Let $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ be a pair in G with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is short. Then (g_1, g_2) is Hurwitz equivalent to a pair of short elements.

Proof. When $\tau_1\tau_2 = a^\epsilon b$, since $Q^{-1}a^\epsilon bQ$ is short, Q is either $(a^\epsilon b)^k a^\zeta$ or $(ba^{-\epsilon})^l a^\zeta$ with $k, l \geq 0$ and $\zeta = 0, 1, 2$. If $Q = a^\zeta$, then both $Q^{-1}\tau_1 Q$ and $Q^{-1}\tau_2 Q$ are short. The result follows from Lemma 2.3.8. When $\tau_1\tau_2 = ba^\epsilon$ or $\tau_1\tau_2 = a^\epsilon ba^\epsilon$, the proof is similar. □

We introduce the following operations and their restorations on an n-tuple $(g_1, \ldots, g_n)$ of elements in G conjugate to some short elements.

- Operation 1 : For $i \in \{1, \ldots, n-1\}$, suppose that the reduced forms of g_i and g_{i+1} are expressed by $Q_i^{-1}\tau_i Q_i$ and $Q_{i+1}^{-1}\tau_{i+1}Q_{i+1}$ with $\tau_i, \tau_{i+1} \in \mathcal{S}$, $Q_i, Q_{i+1} \in G$ such that $Q_i = Q_{i+1}$, (τ_i, τ_{i+1}) is listed in Table 2.1 and either $Q_i = 1$ or $\tau_i\tau_{i+1} = 1$ or both τ_i, τ_{i+1} are powers of a. Then, the operation is a contraction as in Subsection 2.1.2 that replaces (g_i, g_{i+1}) with $g_i g_{i+1}$.
- Operation 2 : For $i \in \{1, \ldots, n\}$, suppose that $g_i = 1$. The operation moves the identical component to the rightmost position via elementary transformations, removes it and reduces $(g_1, \ldots, g_n)$ to an $(n-1)$-tuple.

Operation 1 is a contraction, whose restoration is introduced in Subsection 2.1.1. The restoration of Operation 2 will simply add an identical element on the right side of the tuple. The following proposition shows that, if we use the technique introduced in Subsection 2.1.1 carefully, the resulting tuple is under control.

$g_i \cdot g_{i+1}$ g_{i+1} / g_i	a	a^2	b	a^2b	aba	ba^2	ba	a^2ba^2	ab
a	a^2	1		b			aba	ba^2	a^2b
a^2	1	a		ab	ba	a^2ba^2			b
b			1			a^2	a		
a^2b		a^2ba^2	a^2			a	1		
aba		ab		a				1	
ba^2	b	ba			a				1
ba	ba^2	b		1				a^2	
a^2ba^2	a^2b				1				a^2
ab	aba		a			1	a^2		

TABLE 2.1 – Some pairs (g_i, g_{i+1}) of short elements and the products g_ig_{i+1}.

Proposition 2.3.10. *Let $(g_1, \dots, g_n)$ be an inverse-free n-tuple of elements in G conjugate to some short elements such that $g_1 \cdots g_n = 1$. Suppose that we first apply the following operations successively on $(g_1, \dots, g_n)$:*

i) the elementary transformation R_i, but avoiding that g_i is short and $g_{i-1}^{-1}g_ig_{i+1}$ is long ;

ii) the elementary transformation R_i^{-1}, but avoiding that g_{i+1} is short and $g_ig_{i+1}g_i^{-1}$ is long ;

iii) Operation 1 ;

iv) Operation 2 ;

then apply restorations of Operation 1 and 2 in the reverse order. If all components in the resulting tuple before restorations are short, then the initial tuple is Hurwitz equivalent to the resulting tuple after restorations and further Hurwitz equivalent to a tuple of short elements.

Proof. Lemma 2.1.8 shows that the initial tuple is Hurwitz equivalent to the resulting tuple after all operations and restorations. We suppose that each component is short in the tuple before restorations.

Operation 1 may combine $Q^{-1}\tau_iQ$ and $Q^{-1}\tau_jQ$ into $Q^{-1}\tau_i\tau_jQ$ with $Q \in G$ and $\tau_i, \tau_j \in \mathcal{S}$. By elementary transformations, the product is sent to a conjugate of the form $P^{-1}Q^{-1}\tau_i\tau_jQP$ with some $P \in G$. To restore the operation, it is further rewritten as a pair

$$(P^{-1}Q^{-1}\tau_iQP, P^{-1}Q^{-1}\tau_jQP).$$

Suppose that $P^{-1}Q^{-1}\tau_i\tau_jQP$ is short and (τ_i, τ_j) is listed in Table 2.1.

When $\tau_i\tau_j \in \{a, a^2\}$, the element QP must be a power of a. It is not true that both $P^{-1}Q^{-1}\tau_iQP$ and $P^{-1}Q^{-1}\tau_jQP$ are short in general, as a conjugate of b with a power of a may be long. We list all exceptional possibilities of $(P^{-1}Q^{-1}\tau_iQP, P^{-1}Q^{-1}\tau_jQP)$ as below.

$$(ba^2, aba^2), (aba, a^2ba), (ba, a^2ba), (a^2ba^2, aba^2), (a^2ba, a^2b), (aba^2, aba), (aba^2, ab), (a^2ba, a^2ba^2).$$

However, each of them can be transformed into a pair of short elements by at most two elementary transformations.

When $\tau_i\tau_j = b$, the element QP must be a power of b. A conjugate of a or a^2 with a power of b may be long. The exceptional possibilities of $(P^{-1}Q^{-1}\tau_iQP, P^{-1}Q^{-1}\tau_jQP)$ that one of the components is long are listed as below.

$$(bab, ba^2), (ba^2b, ba), (a^2b, bab), (ab, ba^2b).$$

Again, each of them can be transformed into a pair of short elements by an elementary transformation.

When $\tau_i\tau_j = 1$, we get $P = 1$. Assume that one of $Q^{-1}\tau_iQ$ and $Q^{-1}\tau_jQ$ is long, the inverse-freeness of $(g_1, \dots, g_n)$ implies that τ_i and τ_j are powers of a and Q is not a power of a. Due to the hypothesis that elementary transformations never make short elements long, after all restorations, $Q^{-1}\tau_i\tau_jQ$ (as an additional identical element) will become a sub-tuple $(h_1, \cdots, h_m)$ with $m \geq 2$ such that $h_1, \dots, h_m$ are conjugate to the powers of a simultaneously. It contradicts the inverse-freeness. Thus, both $Q^{-1}\tau_iQ$ and $Q^{-1}\tau_jQ$ are short.

The remaining cases shown in Table 2.1 are covered by Lemma 2.3.9. Hence, by successive application of elementary transformations, each of the components of the resulting tuple is short. □

Definition 2.3.11. The $\mathcal{S}$-*complexity* of an element $g \in G$ conjugate to some element in $\mathcal{S}$ is defined as $f(g)$ such that

$$f(g) = \begin{cases} l(Q) & \text{if } g = Q^{-1}wQ \text{ is long with } w \in \{a^\epsilon, b, a^\epsilon b a^\epsilon | \epsilon = 1,2\} \text{ and } Q \in G\,; \\ 0 & \text{if } g \text{ is short.} \end{cases}$$

Definition 2.3.12. Let $(g_1, \ldots, g_n)$ be an n-tuple in G such that each of g_i, $i = 1, \ldots, n$, is conjugate to some element in $\mathcal{S}$. A sequence of elementary transformations $(R_{i_1}^{\epsilon_1}, \ldots, R_{i_m}^{\epsilon_m})$, $\epsilon_1, \ldots, \epsilon_m \in \{1, -1\}$, is said to make the sum of $\mathcal{S}$-complexities of $(g_1, \ldots, g_n)$ *strictly-smaller* if, for each $m' < m$, the composition $R_{i_{m'}}^{\epsilon_{m'}} \circ \cdots \circ R_{i_1}^{\epsilon_1}$ transforms $(g_1, \ldots, g_n)$ into a tuple with the same sum of $\mathcal{S}$-complexities but $R_{i_m}^{\epsilon_m} \circ \cdots \circ R_{i_1}^{\epsilon_1}$ transforms $(g_1, \ldots, g_n)$ into a tuple with a smaller sum of $\mathcal{S}$-complexities.

A sequence of elementary transformations that makes the sum of $\mathcal{S}$-complexities strictly-smaller never makes short elements long, as described in Proposition 2.3.10 i) and ii).

Let $(g_1, \ldots, g_n)$ be an n-tuple in G. For $i = 1, \ldots, n-1$, suppose that the reduced forms of g_i and g_{i+1} are expressed by $t_{k_i}^{(i)} \ldots t_1^{(i)}$ and $\tilde{t}_1^{(i)} \ldots \tilde{t}_{l_i}^{(i)}$ with $k_i = l(g_i)$, $l_i = l(g_{i+1})$, $t_j^{(i)} \in \{a, a^2, b\}$, $j = 1, \ldots, k_i$ and $\tilde{t}_j^{(i)} \in \{a, a^2, b\}$, $j = 1, \ldots, l_i$. The reduced form of $g_i g_{i+1}$ is then either

$$t_{k_i}^{(i)} \ldots t_{m_i+1}^{(i)} r_i \tilde{t}_{m_i+1}^{(i)} \ldots \tilde{t}_{l_i}^{(i)} \text{ or } t_{k_i}^{(i)} \ldots t_{m_i+1}^{(i)} r_i \text{ or } r_i \tilde{t}_{m_i+1}^{(i)} \ldots \tilde{t}_{l_i}^{(i)}$$

where $r_i \in G$, $l(r_i) \leq 1$ and $0 \leq m_i \leq k_i, l_i$.

Lemma 2.3.13. *Let $(g_1, \ldots, g_n)$ be an n-tuple in G such that each of g_i, $i = 1, \ldots, n$, is conjugate to some element in $\mathcal{S}$ and $g_1 \cdots g_n = 1$. Let m_i be the same as above and set $m_0 = m_n = 0$ for convenience. Suppose that*

(1) there is no pair of adjacent components g_i, g_{i+1} of the reduced forms $Q^{-1}\tau_i Q$, $Q^{-1}\tau_{i+1}Q$ with $Q \in G$ and (τ_i, τ_{i+1}) in Table 2.1 such that either $Q_i = 1$ or $\tau_i\tau_{i+1} = 1$ or both τ_i, τ_{i+1} are powers of a.

(2) there is no sequence of elementary transformations that makes $\sum_i f(g_i)$ strictly-smaller.

Then $m_0, \ldots, m_n$ have the following properties.

(a) For $i = 1, \ldots, n-1$, $m_i \leq \frac{l(g_i)+1}{2}$ and $m_i \leq \frac{l(g_{i+1})+1}{2}$.

(b) For $i = 1, \ldots, n$, $m_{i-1} + m_i \geq l(g_i)$ only if the reduced form of g_i is $Q_i^{-1}a^{\epsilon_i}Q_i$ with $\epsilon_i = 1, 2$, $Q_i \in G$ and $l(Q_i) \geq 0$.

(c) If $m_{i-1} + m_i \leq l(g_i)$ for each of $i = 1, \ldots, n$, then $n = 0$.

Proof. (a) When both g_i and g_{i+1} are short, since (g_i, g_{i+1}) does not figure in Table 2.1, we check all possibilities and get that $m_i \leq \frac{l(g_i)+1}{2}, \frac{l(g_{i+1})+1}{2}$.

When $g_i \in \mathcal{S}$ but $g_{i+1} \notin \mathcal{S}$, say $g_{i+1} = Q_i^{-1}a^{\epsilon_i}Q_i$ or $Q_i^{-1}bQ_i$ or $Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}Q_i$ with $\epsilon_i = 1, 2$ and $l(Q_i) \geq 1$, therefore $l(g_i) \leq 3 \leq l(g_{i+1})$. Assume that $m_i > \frac{l(g_i)+1}{2}$, then $m_i = l(g_i)$ and $l(g_i) \geq 2$. If $l(g_i) = m_i = 2$, as g_{i+1} is long, then $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2). If $l(g_i) = m_i = l(g_{i+1}) = 3$, then the pair (g_i, g_{i+1}) is either (s_1, a^2ba) or (t_1, aba^2), which can be transformed into a pair of short elements by R_1^2, contradicting the hypothesis (2) If $l(g_i) = m_i = 3$ but $l(g_{i+1}) \geq 5$, then again $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, a contradiction. Hence $m_i \leq \frac{l(g_i)+1}{2} \leq \frac{l(g_{i+1})+1}{2}$.

When $g_i \notin \mathcal{S}$ but $g_{i+1} \in \mathcal{S}$, there is a similar argument.

When both g_i and g_{i+1} are long, suppose that their reduced forms are

$$Q_i^{-1}w_iQ_i \text{ and } Q_{i+1}^{-1}w_{i+1}Q_{i+1}$$

with $w_i, w_{i+1} \in \{a, a^2, b, aba, a^2ba^2\}$. Assume that $l(Q_i) \leq l(Q_{i+1})$ without loss of generality. Assume that $m_i > \min\{\frac{l(g_i)+1}{2}, \frac{l(g_{i+1})+1}{2}\}$. Therefore Q_{i+1} ends with Q_i. Write $Q_{i+1} = \tilde{Q}Q_{i+1}$ and

$$(g_i, g_{i+1}) = (Q_i^{-1}w_iQ_i, Q_i^{-1}\tilde{Q}^{-1}w_{i+1}\tilde{Q}Q_i).$$

Suppose that $l(\tilde{Q}) = 0$. We further assume that $l(w_i) \leq l(w_{i+1})$ without loss of generality. Since $m_i > l(Q_i) + \frac{l(w_i)+1}{2}$, $w_iw_{i+1} \neq 1$ and one of w_i, w_{i+1} is not a power of a, the pair (w_i, w_{i+1}) must be either (a, a^2ba^2) or (a^2, aba). Therefore, $l(g_ig_{i+1}g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2).

Suppose that the element $\tilde{Q}$ is of length at least 1. Therefore $l(w_i) \leq 3 \leq l(\tilde{Q}^{-1}w_{i+1}\tilde{Q})$ and $m_i > \frac{l(g_i)+1}{2}$. If $l(w_i) = 1$, then $m_i > l(Q_i)+1$ and $l(g_ig_{i+1}g_i^{-1}) \leq l(g_{i+1})-2$, contradicting the hypothesis

(2). If $l(w_i)=3$ and $l(\tilde{Q}^{-1}w_{i+1}\tilde{Q})=3$, then $(w_i,\tilde{Q}^{-1}w_{i+1}\tilde{Q})$ is either (s_1,a^2ba) or (t_1,aba^2) and therefore (g_i,g_{i+1}) can be transformed into either $(Q_i^{-1}a^2bQ_i,Q_i^{-1}bQ_i)$ or $(Q_i^{-1}abQ_i,Q_i^{-1}bQ_i)$ by R_1^2, contradicting the hypothesis (2). If $l(w_i)=3$ and $l(\tilde{Q}^{-1}w_{i+1}\tilde{Q})\geq 5$, then $m_i\geq l(Q_i)+3$ and $l(g_ig_{i+1}g_i^{-1})\leq l(g_{i+1})-2$, contradicting the hypothesis (2).

(b) Suppose that $m_{i-1}+m_i\geq l(g_i)$ for some $i=1,\ldots,n-1$.

Suppose that $g_i\in\mathcal{S}$. If g_i is of length 2 (i.e. the element g_i is one of s_0, s_2, t_0 and t_2), then $m_{i-1}=m_i=1$. Therefore, one of g_{i-1}, g_{i+1} is equal to b, contradicting Table 2.1. If $g_i=b$, then one of g_{i-1} and g_{i+1} is long starting and ending with b. Therefore, either $l(g_i^{-1}g_{i-1}g_i)<l(g_{i-1})$ or $l(g_ig_{i+1}g_i^{-1})<l(g_{i+1})$, contradicting the hypothesis (2). If $g_i=a^{\epsilon_i}ba^{\epsilon_i}$ with $\epsilon_i=1,2$, then one of m_{i-1} and m_i is equal to 2 and thus one of g_{i-1}, g_{i+1} is long. It is impossible as long elements are of length at least 3.

Suppose that g_i is long. If $g_i=Q_i^{-1}bQ_i$, then either $g_{i-1}=bQ_i$ or $g_{i+1}=Q_i^{-1}b$, but thus either $g_{i-1}g_ig_{i-1}^{-1}=bQ_i(Q_i^{-1}bQ_i)Q_i^{-1}b=b$ or $g_{i+1}^{-1}g_ig_{i+1}=bQ_i(Q_i^{-1}bQ_i)Q_i^{-1}b=b$. If $g_i=Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}Q_i$ with $\epsilon_i=1,2$, then either $m_{i-1}=l(Q_i^{-1}a^{\epsilon_i}b)$ or $m_i=l(ba^{\epsilon_i}Q_i)$. It implies that either $g_{i-1}=ba^{-\epsilon_i}Q_i$ or $g_{i+1}=Q_i^{-1}a^{-\epsilon_i}b$, thus either $g_{i-1}g_ig_{i-1}^{-1}=a^{2\epsilon_i}b$ or $g_{i+1}^{-1}g_ig_{i+1}=ba^{2\epsilon_i}$. Both cases contradict the hypothesis (2).

We conclude that either $g_i=a^{\epsilon_i}$ or $g_i=Q_i^{-1}a^{\epsilon_i}Q_i$.

(c) Assume that $n\geq 1$.

By (2), when $m_{i-1}+m_i=l(g_i)$, then $g_i=Q_i^{-1}a^{\epsilon_i}Q_i$ with $\epsilon_i=1,2$ and $l(Q_i)\geq 0$. If $g_i=a^{\epsilon_i}$ is short and assume that $m_{i-1}=0$, $m_i=1$ without loss of generality, then g_{i+1} is either $a^{\epsilon_i}ba^{\epsilon_i}$ or a long element starting with a^{ϵ_i}. If $g_i=Q_i^{-1}a^{\epsilon_i}Q_i$ is long and $m_{i-1}=l(Q_i)$, to avoid $l(g_{i-1}g_ig_{i-1}^{-1})<l(g_i)$ then g_{i-1} must be longer than Q_i^{-1}, contradicting the hypothesis that $m_{i-1}=l(Q_i)$. Therefore, neither m_{i-1} nor m_i is equal to $l(Q_i)$ and in particular, $m_{i-1}+m_i<l(g_i)$.

The proof of (2) and the above observation show that there is no possibility to fully reduce g_i or g_{i+1} in the product g_ig_{i+1} and $m_{i-1}+m_i<l(g_i)$ if $g_i\neq a^{\epsilon_i}$. They imply a contradiction that $g_1\cdots g_n\neq 1$. □

Now we introduce the main result in this subsection.

Theorem 2.3.14. *Let $g_1,\ldots,g_n$ be such that each of them is conjugate to some element in $\mathcal{S}$ and $g_1\cdots g_n=1$. Then, the n-tuple $(g_1,\ldots,g_n)$ is Hurwitz equivalent to either*

- $(h_1,\ldots,h_\mu)\bullet(s_0,t_0)^{m_{st}}\bullet(a,a^2)^{m_a}\bullet(b,b)^{m_b}\bullet(a,a,a)^{n_0}\bullet(a^2,a^2,a^2)^{n_1}$ *with* $\mu>0$, m_{st}, m_a, m_b, n_0, $n_1\geq 0$, $\mu+2(m_{st}+m_a+m_b)+3(n_0+n_1)=n$ *such that* $(h_1,\ldots,h_\mu)$ *is an inverse-free* μ*-tuple of short elements, or*
- $(k_1,k_1^{-1},\ldots,k_s,k_s^{-1},l_1,l_1,l_1,\ldots,l_t,l_t,l_t)$ *with* $s,t\geq 0$, $2s+3t=n$, $k_1,\ldots,k_s\in G$, $l_1,\ldots,l_t\in G$ *and* $l_j^3=1$ *for each* $j=1,\ldots,t$.

Proof. We first attempt to make the tuple inverse-free. Applying any finite sequence of elementary transformations to $(g_1,\ldots,g_n)$, if we get a pair of mutually inverse elements or a triple of the form (l,l,l) with $l\in G$ and $l^3=1$, then we move it to the rightmost position via elementary transformations and the resulting tuple is the concatenation of a shorter tuple and either a pair or a triple. By induction on the length, we suppose that the n-tuple $(g_1,\ldots,g_n)$ is transformed into the concatenation of $(h_1,\ldots,h_\mu)$ and $(k_1,k_1^{-1},\ldots,k_s,k_s^{-1},l_1,l_1,l_1,\ldots,l_t,l_t,l_t)$ with $\mu,s,t\geq 0$, $\mu+2s+3t=n$ such that $l_j^3=1$ for each $j=1,\ldots,t$ where $(h_1,\ldots,h_\mu)$ is inverse-free.

We will always use the notation m_i to indicate the length of the reduced part in h_ih_{i+1} for $i=1,\ldots,\mu-1$ and set $m_0=m_\mu=0$ as before.

To prove the theorem for $(h_1,\ldots,h_\mu)$, we use induction on

$$\Big(\mu,\sum_{i=1}^{\mu}f(h_i),l(h_1),\ldots,l(h_\mu)\Big)$$

and apply the following operations on $(h_1,\ldots,h_\mu)$. If there exists a pair of adjacent components which has the reduced form $(Q^{-1}\tau_iQ,Q^{-1}\tau_jQ)$ with $Q\in G$, (τ_i,τ_j) in Table 2.1 such that either $Q=1$ or $\tau_i\tau_j=1$ or both τ_i, τ_j are powers of a, then we replace them with their product and reduce $(h_1,\ldots,h_\mu)$ to a $(\mu-1)$-tuple. If there exists an identical component, then we move it to the rightmost position and remove it. If there exists a proper sequence of elementary transformations that can make $\sum_i f(h_i)$ strictly-smaller, then we apply it.

When each of the above operations fails, the resulting tuple, still denoted by $(h_1,\ldots,h_\mu)$, satisfies all hypotheses in Lemma 2.3.13. Suppose that $\mu\geq 1$ and there exists some $i=2,\ldots,\mu-1$ such that $h_i=Q_i^{-1}a^{\epsilon_i}Q$ with $\epsilon_i=1,2$, $l(Q_i)\geq 0$ and $m_{i-1}=m_i=l(Q_i)+1$. Then, either

- $(h_{i-1},h_i)=(a^{\epsilon_i}ba^{\epsilon_i},a^{\epsilon_i})$, or

- the previous component h_{i-1} is long and $l(h_{i-1}) \geq l(h_i)$.

In the second case, we first assume that $l(h_{i-1}) = l(h_i)$. Then $h_{i-1} = Q_i^{-1} a^{\epsilon_i} Q_i = h_i$, which is a contradiction. Hence, $l(h_{i-1}) > l(h_i)$ and, to avoid $l(h_i^{-1} h_{i-1} h_i) < l(h_{i-1})$, we claim that h_{i-1} must end with $a^{\epsilon_i} Q_i$ and start with $Q_i^{-1} a^{-\epsilon_i}$. In both cases, $l(h_i^{-1} h_{i-1} h_i) \leq l(h_{i-1})$ and we are able to reduce $(h_1, \ldots, h_\mu)$ to an n-tuple, say $(\tilde{h}_1, \ldots, \tilde{h}_\mu)$, such that $\sum_j f(h_j) = \sum_j f(\tilde{h}_j)$, $l(h_j) = l(\tilde{h}_j)$ for $1 \leq j < i-1$ but $l(\tilde{h}_{i-1}) = l(h_i) < l(h_{i-1})$ via the elementary transformation R_{i-1}.

The induction does not stop unless μ is equal to 0. Due to Proposition 2.3.10, by restoring the operations and applying more elementary transformations, we get a resulting μ-tuple of short elements that can be obtained from the original $(h_1, \ldots, h_\mu)$ via elementary transformations directly. Hence, the n-tuple $(g_1, \ldots, g_n)$ can be transformed into

$$(h'_1, \ldots, h'_\mu, k_1, k_1^{-1}, \ldots, k_s, k_s^{-1}, l_1, l_1, l_1, \ldots, l_t, l_t, l_t)$$

with $\mu, s, t \geq 0$, $\mu + 2s + 3t = n$, $l_j^3 = 1$ for each $j = 1, \ldots, t$ such that each of h'_i, $i = 1, \ldots, \mu$ is short and $(h'_1, \ldots, h'_\mu)$ is inverse-free.

Suppose that $\mu > 0$. There is always a pair (h'_i, h'_j) of components with $1 \leq i \neq j \leq \mu$ that is a generating set of G. By Lemma 2.1.6, each pair of the form $(k, k^{-1}) = (Q^{-1}wQ, Q^{-1}w^{-1}Q)$ with $Q \in G$, $w, w^{-1} \in \{a, a^2, b, s_0\}$ in the resulting tuple can be transformed into (w, w^{-1}). There is a similar argument for each triple (l, l, l) with $l^3 = 1$. Hence, by elementary transformations, the n-tuple can be transformed into a tuple of short elements. □

Theorem 2.3.14 is surprising. In fact, there are infinitely many pairs (g_s, g_t) in G up to Hurwitz equivalence such that $g_s g_t = 1$ and g_s, g_t are conjugates of s_0 and t_0 respectively. However, all triples (g_a, g_b, g_s) that $g_a g_b g_s = 1$ and g_a, g_b, g_s are conjugates of a, b, s_0 respectively, are mutually Hurwitz equivalent. In particular, for any $Q \in G$, we have

$$(Q^{-1}aQ, Q^{-1}aba^2Q, Q^{-1}abaQ) \sim (a, b, s_2).$$

2.3.3 Classification of tuples up to Hurwitz equivalence

Given $g_1, \ldots, g_n$ and $h_1, \ldots, h_n \in G$ conjugate to elements in $\mathcal{S}$ such that $g_1 \cdots g_n = h_1 \cdots h_n = 1$, suppose that the n-tuples $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ have the same number of components in each conjugacy class. In this subsection, we show that $(g_1, \ldots, g_n)$ is Hurwitz equivalent to $(h_1, \ldots, h_n)$ in most cases. In particular, we introduce a normal form for tuples of elements conjugate to some short elements that only depends on the numbers of components in every conjugacy classes.

The following theorem is a partial result, which interprets the projective global monodromy of an achiral Lefschetz fibration. Matsumoto presented a slightly different theorem in [Mat85, Theorem 3.6].

Theorem 2.3.15. *Let $g_1, \ldots, g_n \in G$ be such that p of them are conjugates of s_0, $q = n - p$ of them are conjugates of t_0 and $g_1 \cdots g_n = 1$. Then,*

1. *if $p > q$, then $p - q \equiv 0 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to $(s_0, s_2)^{(p-q)/2} \bullet (s_0, t_0)^q$;*
2. *if $p < q$, then $q - p \equiv 0 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to $(t_0, t_2)^{(q-p)/2} \bullet (s_0, t_0)^p$;*
3. *if $p = q$, then the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to $(k_1, k_1^{-1}, \ldots, k_p, k_p^{-1})$ where each of k_j, $j = 1, \ldots, p$, is conjugate to s_0.*

Proof. Theorem 2.3.14 reveals that, by elementary transformations, the n-tuple can be transformed into either $(k_1, k_1^{-1}, \ldots, k_s, k_s^{-1})$ with $s = p = q$ or $(h_1, \ldots, h_\mu) \bullet (s_0, t_0)^{m_{st}}$ with $\mu > 0$, $m_{st} \geq 0$, $\mu + 2m_{st} = n$ such that $(h_1, \ldots, h_\mu)$ is an inverse-free μ-tuple of short elements. On the latter, by Lemma 2.3.5, we get $\mu \equiv 0 \pmod 6$ and $(h_1, \ldots, h_\mu)$ can be transformed into either $(s_0, s_2)^{\mu/2}$ or $(t_0, t_2)^{\mu/2}$ by elementary transformations. Hence, $m_{st} = \min\{p, q\}$ and $\mu = |p - q|$. □

In general, we have Theorem 2.1.16, whose proof will be given at the end.

Lemma 2.3.16. *Let $g_1, \ldots, g_n$ be a^2, s_0, s_1 or s_2 such that only one of them is equal to a^2 and $g_1 \cdots g_n = 1$. Then, $n \equiv 3 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$(a^2, s_0, s_2) \bullet (s_0, s_2)^{(n-3)/2}.$$

Proof. Since a cyclic permutation of an n-tuple in G can be obtained by a finite sequence of elementary transformations as in Lemma 2.1.2, we may assume that $g_1 = a^2$ without loss of generality. Since $g_1 \cdots g_n = 1$, then $n \geq 3$. If $n = 3$, then the pair $(g_1 g_2, g_3)$ must be equal to (t_j, s_j) with some j and therefore (g_1, g_2, g_3) is given by (a^2, s_{j+1}, s_j) as $at_j = s_{j+1}$. Otherwise, $n \geq 4$. We replace g_1 and g_2 with their product, which is one of t_0, t_1 and t_2. The n-tuple $(g_1, \ldots, g_n)$ is replaced by an $(n-1)$-tuple whose first component belongs to $\{t_0, t_1, t_2\}$ and the rest components are s_0, s_1 or s_2. By Theorem 2.3.15, $(n-1) - 2 \equiv 0 \pmod 6$ and the $(n-1)$-tuple can be transformed into $(s_0, s_2)^{(n-3)/2} \bullet (s_0, t_0)$ by successive application of elementary transformations. We note that the original n-tuple $(g_1, \ldots, g_n)$ must be inverse-free and, after combining s_0 and t_0 into 1 and removing it, we make the result an inverse-free tuple of short elements. By Proposition 2.3.10, it implies a sequence of elementary transformations sending the n-tuple $(g_1, \ldots, g_n)$ into $(s_0, a^2, s_1) \bullet (s_0, s_2)^{(n-3)/2}$. In any case, the substitutions of (s_0, s_2), (s_1, s_0) and (s_2, s_1) transform the n-tuple $(g_1, \ldots, g_n)$ into $(a^2, s_0, s_2) \bullet (s_0, s_2)^{(n-3)/2}$. □

The following lemma can be proved similarly and we omit the details.

Lemma 2.3.17. *Let $g_1, \ldots, g_n$ be a, t_0, t_1 or t_2 such that only one of them is equal to a and $g_1 \cdots g_n = 1$. Then, $n \equiv 3 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$(a, t_2, t_0) \bullet (t_0, t_2)^{(n-3)/2}.$$

Lemma 2.3.18. *Let $g_1, \ldots, g_n$ be b, s_0, s_1 or s_2 such that only one of them is equal to b and $g_1 \cdots g_n = 1$. Then, $n \equiv 4 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$(b, s_0, s_2, s_0) \bullet (s_0, s_2)^{(n-4)/2}.$$

Proof. Without loss of generality, we assume that all the n-tuple in $\{b, s_0, s_1, s_2\}$ resulting from the successive application of elementary transformations on $(g_1, \ldots, g_n)$ contain no consecutive sub-tuples of the form $(s_0, s_2)^3$.

Take the n-tuple in $\{b, s_0, s_1, s_2\}$ that starts with b and contains the minimal number of components equal to s_1 among all resulting tuples that we can get using elementary transformations on $(g_1, \ldots, g_n)$, still denoted by $(g_1, \ldots, g_n)$. We write it as

$$(b) \bullet \Big(\prod_{j=1}^{n_0} (s_2)^{u_{0,j}} \bullet (s_0)^{v_{0,j}} \Big) \bullet \Big(\prod_{i=1}^{\mu} (s_1)^{\lambda_i} \bullet \Big(\prod_{j=1}^{n_i} (s_2)^{u_{i,j}} \bullet (s_0)^{v_{i,j}} \Big) \Big)$$

with $\mu \geq 0$, $\lambda_1, \ldots, \lambda_\mu \geq 1$, $n_0, n_\mu \geq 0$, $n_1, \ldots, n_{\mu-1} \geq 1$ and $u_{i,j}$, $v_{i,j} \geq 0$ for $i = 0, \ldots, \mu$, $j = 1, \ldots, n_i$ where $u_{i,j} \geq 1$ for $j > 1$ and $v_{i,j} \geq 1$ for $j < n_i$. The minimality further requires that $u_{i,1} \geq 1$ for $i = 1, \ldots, \mu$ and $v_{i,n_i} \geq 1$ for $i = 0, \ldots, \mu - 1$.

Assume that $(g_1, \ldots, g_n)$ does not start with (b, s_2) or $(b) \bullet (s_0)^{v_{0,1}} \bullet (s_2, s_0)$, nor end with s_0 or $(s_2, s_0) \bullet (s_2)^{u_{\mu,n_\mu}}$. Then, $u_{0,1} = v_{\mu,n_\mu} = 0$, either $n_0 \leq 1$ or $u_{0,2} \geq 2$ and either $n_\mu \leq 1$ or $v_{\mu,n_\mu-1} \geq 2$. Applying Proposition 2.3.6 with the above restrictions, we obtain that the reduced form of $g_1 \cdots g_n$ is not equal to 1, which is a contradiction. Hence, using the substitution of (s_0, s_2, s_0) and (s_2, s_0, s_2) and a cyclic permutation if necessary, the n-tuple is transformed into an n-tuple in $\{b, s_0, s_1, s_2\}$, still denoted by $(g_1, \ldots, g_n)$, such that (g_1, g_2) is equal to either (b, s_2) or (s_0, b). We combine g_1 and g_2 into their product and replace $(g_1, \ldots, g_n)$ with an $(n-1)$-tuple in $\{a^2, s_0, s_1, s_2\}$ starting with a^2. By Lemma 2.3.16 we get $n - 1 \equiv 3 \pmod 6$ and, by successive application of elementary transformations, the $(n-1)$-tuple can be transformed into $(a^2, s_0, s_2) \bullet (s_0, s_2)^{(n-4)/2}$. By Proposition 2.3.10, we obtain an n-tuple of the form either $(b, s_2, s_0, s_2) \bullet (s_0, s_2)^{(n-4)/2}$ or $(s_0, b, s_0, s_2) \bullet (s_0, s_2)^{(n-4)/2}$ from $(g_1, \ldots, g_n)$ using elementary transformations. The substitution of (s_0, s_2, s_0) and (s_2, s_0, s_2) completes the proof. □

Again, the following lemma is similar and we omit the proof.

Lemma 2.3.19. *Let $g_1, \ldots, g_n$ be b, t_0, t_1 or t_2 such that only one of them is equal to b and $g_1 \cdots g_n = 1$. Then, $n \equiv 4 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$(b, t_0, t_2, t_0) \bullet (t_0, t_2)^{(n-4)/2}.$$

Lemma 2.3.20. *Let $g_1, \ldots, g_n$ be a, s_0, s_1 or s_2 such that only one of them is equal to a and $g_1 \cdots g_n = 1$. Then, $n \equiv 5 \pmod 6$ and the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$(a, s_0, s_0, s_2, s_0) \bullet (s_0, s_2)^{(n-5)/2}.$$

Proof. Without loss of generality, we assume that each n-tuple in $\{a, s_0, s_1, s_2\}$ that results from the successive application of elementary transformations on $(g_1, \dots, g_n)$ contains no consecutive sub-tuples of the form $(s_0, s_2)^3$.

Take the n-tuple in $\{a, s_0, s_1, s_2\}$ that starts with a and contains the minimal number of components equal to s_1 among all resulting tuples that we can get using elementary transformations on $(g_1, \dots, g_n)$, still denoted by $(g_1, \dots, g_n)$. Assume that

$$(g_1, g_2) \neq (a, s_0), (g_n, g_1) \neq (s_2, a) \text{ and } (g_n, g_1, g_2) \neq (s_1, a, s_1).$$

Then, the n-tuple $(g_1, \dots, g_n)$ is written as

$$(a) \bullet \Big(\prod_{j=1}^{n_0} (s_2)^{u_{0,j}} \bullet (s_0)^{v_{0,j}} \Big) \bullet \Big(\prod_{i=1}^{\mu} (s_1)^{\lambda_i} \bullet \Big(\prod_{j=1}^{n_i} (s_2)^{u_{i,j}} \bullet (s_0)^{v_{i,j}} \Big) \Big)$$

with $\mu \geq 0$, $\lambda_1, \dots, \lambda_\mu \geq 1$, $n_0, n_\mu \geq 0$ but $n_0 + n_\mu \geq 1$, $n_1, \dots, n_{\mu-1} \geq 1$ and $u_{i,j}, v_{i,j} \geq 1$ for $i = 0, \dots, \mu$, $j = 1, \dots, n_i$. Applying Proposition 2.3.6, we notice that the reduced form of $g_1 \cdots g_n$ is not equal to 1, which is a contradiction. Hence, one of the above requirements cannot be fulfilled.

If either $(g_1, g_2) = (a, s_0)$ or $(g_n, g_1) = (s_2, a)$, using a cyclic permutation if necessary, then the pair (g_1, g_2) is equal to either (a, s_0) or (s_2, a). We combine g_1 and g_2 into a single b and replace $(g_1, \dots, g_n)$ with an $(n-1)$-tuple in $\{b, s_0, s_1, s_2\}$ starting with the only b. By Lemma 2.3.18, we get $n-1 \equiv 4 \pmod 6$ and there exists a finite sequence of elementary transformations that transforms the $(n-1)$-tuple into $(b, s_0, s_2, s_0) \bullet (s_0, s_2)^{(n-5)/2}$. Proposition 2.3.10 implies that $(g_1, \dots, g_n)$ can be transformed by elementary transformations into either

$$(a, s_0, s_0, s_2, s_0) \bullet (s_0, s_2)^{(n-5)/2} \text{ or } (s_2, a, s_0, s_2, s_0) \bullet (s_0, s_2)^{(n-5)/2}.$$

If $(g_n, g_1, g_2) = (s_1, a, s_1)$, by a cyclic permutation and an elementary transformation, then the n-tuple can be transformed into $(s_1, s_0, a, g_3, \dots, g_{n-1})$. We combine s_1 and s_0 into a, further combine a and a into a single a^2 and replace the n-tuple with an $(n-2)$-tuple in $\{a^2, s_0, s_1, s_2\}$ starting with the only a^2. By Lemma 2.3.16, we get $n - 2 \equiv 3 \pmod 6$ and the $(n-2)$-tuple can be transformed by elementary transformations into $(a^2, s_0, s_2) \bullet (s_0, s_2)^{(n-5)/2}$. By Proposition 2.3.10, the n-tuple $(g_1, \dots, g_n)$ can be transformed into $(s_{j+1}, s_j, a, s_0, s_2) \bullet (s_0, s_2)^{(n-5)/2}$ with some j. The substitutions of (s_0, s_2), (s_1, s_0), (s_2, s_1) and the substitution of (s_0, s_2, s_0), (s_2, s_0, s_2) conclude the lemma. $\square$

Once again, the following lemma is similar and we omit the proof.

Lemma 2.3.21. *Let $g_1, \dots, g_n$ be a^2, t_0, t_1 or t_2 such that only one of them is equal to a^2 and $g_1 \cdots g_n = 1$. Then, $n \equiv 5 \pmod 6$ and the n-tuple $(g_1, \dots, g_n)$ is Hurwitz equivalent to*

$$(a^2, t_0, t_2, t_0, t_0) \bullet (t_0, t_2)^{(n-5)/2}.$$

Proof of Theorem 2.1.16. By Theorem 2.3.14, we are able to transform $(g_1, \dots, g_n)$ into either

$$(k_1, k_1^{-1}, \dots, k_s, k_s^{-1}, l_1, l_1, l_1, \dots, l_t, l_t, l_t)$$

with $s, t \geq 0$, $2s + 3t = n$ and $l_j^3 = 1$ for each $j = 1, \dots, t$, or

$$(h_1, \dots, h_\mu) \bullet (s_0, t_0)^{m_{st}} \bullet (a, a^2)^{m_a} \bullet (b, b)^{m_b} \bullet (a, a, a)^{n_0} \bullet (a^2, a^2, a^2)^{n_1}$$

with $\mu > 0$, m_{st}, m_a, m_b, n_0, $n_1 \geq 0$, $\mu + 2(m_{st} + m_a + m_b) + 3(n_0 + n_1) = n$ such that $(h_1, \dots, h_\mu)$ is an inverse-free μ-tuple of short elements. The former case just so happens to be the first case of Theorem 2.1.16, therefore we consider only the latter and suppose that $\mu > 0$. As $(h_1, \dots, h_\mu)$ is inverse-free, it contains at most two a's, at most two a^2's, at most one b and it does not contain both a and a^2. Let $\mathcal{A}$ be the set of elements in $(h_1, \dots, h_\mu)$. Let I_a and I_b be the numbers of components conjugate to some power of a and b respectively. Take $\mathcal{A}_s = \mathcal{A} \cap \{s_0, s_1, s_2\}$ and $\mathcal{A}_t = \mathcal{A} \cap \{t_0, t_1, t_2\}$.

Step 1. Suppose that $I_a + I_b \leq 1$. Proposition 2.3.4 shows that either $\mathcal{A}_s$ or $\mathcal{A}_t$ is empty. Thus, by Lemma 2.3.5, 2.3.20, 2.3.17, 2.3.16, 2.3.21, 2.3.18 and 2.3.19, by elementary transformations the inverse-free tuple $(h_1, \dots, h_\mu)$ can be transformed into one of the following partial normal forms.

- $(s_0, s_2, s_0, s_2, s_0, s_2)^{\mu/6}$, $(t_0, t_2, t_0, t_2, t_0, t_2)^{\mu/6}$ where $\mu \equiv 0 \pmod 6$;
- $(a^2, s_0, s_2) \bullet (s_0, s_2)^{(\mu-3)/2}$, $(a, t_2, t_0) \bullet (t_0, t_2)^{(\mu-3)/2}$ where $\mu \equiv 3 \pmod 6$;
- $(b, s_0, s_2, s_0) \bullet (s_0, s_2)^{(\mu-4)/2}$, $(b, t_0, t_2, t_0) \bullet (t_0, t_2)^{(\mu-4)/2}$ where $\mu \equiv 4 \pmod 6$;
- $(a, s_0, s_0, s_2, s_0) \bullet (s_0, s_2)^{(\mu-5)/2}$, $(a^2, t_0, t_2, t_0, t_0) \bullet (t_0, t_2)^{(\mu-5)/2}$ where $\mu \equiv 5 \pmod 6$.

Step 2. Suppose that $I_a = 1 = I_b$ and $a^\epsilon \in \mathcal{A}$ with $\epsilon = \pm 1$. It is clear that $\mu \geq 3$.

If there exists an element $h' \in \mathcal{A}$ equal to one of ba^ϵ, $a^\epsilon b$, $a^{-\epsilon}ba^{-\epsilon}$ then, using elementary transformations, we place a^ϵ and h' in adjacent positions that form a pair (a^ϵ, h'). The pair is further replaced by the product $a^\epsilon h'$ and we replace $(h_1, \ldots, h_\mu)$ with a $(\mu-1)$-tuple, say $(y_1, \ldots, y_{\mu-1})$. Each of $y_1, \ldots, y_{\mu-1}$ is short, one of them is equal to b and each of the rest is neither a power of a nor b. By Theorem 2.3.14, Proposition 2.3.4 and Lemma 2.3.18, 2.3.19, the $(\mu-1)$-tuple $(y_1, \ldots, y_{\mu-1})$ can be transformed into either

$$(b, s_0, s_2, s_0) \bullet (s_0, s_2, s_0, s_2, s_0, s_2)^u \bullet (s_0, t_0)^v \text{ or } (b, t_0, t_2, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2)^u \bullet (s_0, t_0)^v$$

with $u, v \geq 0$ and $5 + 6u + 2v = \mu$. Proposition 2.3.10 shows that $(h_1, \ldots, h_\mu)$ can be transformed into one of them with exactly one of the following adjustments : replace an s_0 (resp. s_2, t_0, t_2) with (a, t_2) (resp. (a, t_1), (a^2, s_1), (a^2, s_0)). The substitutions

$$\begin{aligned}
&(b, a, t_2, s_2, s_0) \to (b, t_2, s_2, a, s_0) \to (b, a, t_0, s_0, s_0) \to (b, a, s_0) \bullet (t_0, s_0) \to (a, b, s_2) \bullet (s_0, t_0),\\
&(b, s_0, a, t_1, s_0) \to (b, s_0, t_0, a, s_0) \to (b, a, s_0) \bullet (s_0, t_0) \to (a, b, s_2) \bullet (s_0, t_0),\\
&(b, s_0, s_2, a, t_2) \to (b, s_2, s_1, a, t_2) \to (b, s_2, s_1, t_1, a) \to (b, s_2, s_0, t_0, a) \to (a, b, s_2) \bullet (s_0, t_0),\\
&(b, s_0, s_2, s_0) \bullet (a, t_2, s_2, s_0, s_2, s_0, s_2) \to (b, s_0, s_2, a, t_2) \bullet (s_0, s_2, s_0, s_2, s_0, s_2),\\
&(b, s_0, s_2, s_0) \bullet (s_0, a, t_1, s_0, s_2, s_0, s_2) \to (s_0, b, s_0, a, t_1) \bullet (s_0, s_2, s_0, s_2, s_0, s_2),\\
&(b, s_0, s_2, s_0) \bullet (a, t_2, t_0) \to (b, s_0, s_2, s_0) \bullet (t_0, a, t_2) \to (b, s_0, s_2, a, t_2) \bullet (s_0, t_0)
\end{aligned}$$

and their symmetrical manners further transform the resulting μ-tuple into one of the following partial normal forms.

- $(a, b, s_2) \bullet (s_0, s_2, s_0, s_2, s_0, s_2)^u \bullet (s_0, t_0)^{v+1}$ with $u, v \geq 0$;
- $(a, t_2, t_0) \bullet (b, t_0, t_2, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2)^u \bullet (s_0, t_0)^{v-1}$ with $u \geq 0$ and $v \geq 1$;
- $(a^2, b, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2)^u \bullet (s_0, t_0)^{v+1}$ with $u, v \geq 0$;
- $(a^2, s_0, s_2) \bullet (b, s_0, s_2, s_0) \bullet (s_0, s_2, s_0, s_2, s_0, s_2)^u \bullet (s_0, t_0)^{v-1}$ with $u \geq 0$ and $v \geq 1$.

Otherwise, one of $\mathcal{A}_s$ and $\mathcal{A}_t$ is empty. If there exists an element $h' \in \mathcal{A}$ equal to either $ba^{-\epsilon}$ or $a^{-\epsilon}b$ then, using elementary transformations, we place a^ϵ and h' in adjacent positions such that their product is equal to b. The pair is further replaced by a single b. Therefore, the resulting $(\mu-1)$-tuple has exactly two different components conjugate to b and the rest are either conjugates of s_0 or conjugates of t_0. Applying Theorem 2.3.14, we have shown in Step 1 that such an $(\mu-1)$-tuple can be transformed by elementary transformations into either $(s_0, s_2, s_0, s_2, s_0, s_2)^{(\mu-3)/6} \bullet (b, b)$ or $(t_0, t_2, t_0, t_2, t_0, t_2)^{(\mu-3)/6} \bullet (b, b)$. Proposition 2.3.10 implies that $(h_1, \ldots, h_\mu)$ can be transformed into a concatenation of either $(s_0, s_2)^{(\mu-3)/2}$ or $(t_0, t_2)^{(\mu-3)/2}$ and one of the following triples, which can be further transformed into a result consistent with the previous case.

$$(a, s_0, b), (a^2, t_2, b), (s_2, a, b), (t_0, a^2, b), (b, a, s_0), (b, a^2, t_2), (b, s_2, a), (b, t_0, a^2).$$

Step 3. We consider the last case left in Step 2 where $\mathcal{A} = \{a^\epsilon, b, a^\epsilon ba^\epsilon\}$ and $I_a = 1$.

In fact, we have $\mu \geq 4$. By elementary transformations we place a^ϵ and two different $a^\epsilon ba^\epsilon$'s in adjacent positions that form a triple of the form $(a^\epsilon ba^\epsilon, a^\epsilon, a^\epsilon ba^\epsilon)$. The triple can be further transformed into $(a^\epsilon ba^\epsilon, a^{-\epsilon}b, a^\epsilon)$. We combine the first two components into a^ϵ and then rewrite the triple as a single $a^{-\epsilon}$. The resulting $(\mu-2)$-tuple is composed of $a^{-\epsilon}, b$ and several $a^\epsilon ba^\epsilon$. Step 2 has shown that such a tuple can be transformed by elementary transformations into either

$$(a, t_2, t_0) \bullet (b, t_0, t_2, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2)^u \text{ or } (a^2, s_0, s_2) \bullet (b, s_0, s_2, s_0) \bullet (s_0, s_2, s_0, s_2, s_0, s_2)^u$$

with $u \geq 0$. By Proposition 2.3.10, elementary transformations can transform $(h_1, \ldots, h_\mu)$ into either

$$\begin{aligned}
&(t_j, t_{j+1}, a^2, t_2, t_0) \bullet (b, t_0, t_2, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2)^u\\
\text{or } &(s_j, s_{j-1}, a, s_0, s_2) \bullet (b, s_0, s_2, s_0) \bullet (s_0, s_2, s_0, s_2, s_0, s_2)^u
\end{aligned}$$

that can be further transformed into the result in Step 2 using the substitutions :

$$\begin{aligned}
&(t_j, t_{j+1}, a^2, t_2, t_0) \bullet (b, t_0, t_2, t_0) \to (a^2, t_2, t_0, t_2, t_0) \bullet (t_0, t_2, t_0, b) \to (a^2, b, t_0) \bullet (t_0, t_2, t_0, t_2, t_0, t_2),\\
&(s_j, s_{j-1}, a, s_0, s_2) \bullet (b, s_0, s_2, s_0) \to (a, s_0, s_2, s_0, s_2) \bullet (s_0, s_2, s_0, b) \to (a, b, s_2) \bullet (s_0, s_2, s_0, s_2, s_0, s_2).
\end{aligned}$$

Step 4. Suppose that $I_a = 2$.

We place the powers of a in adjacent positions and replace them with their product. The resulting $(\mu-1)$-tuple contains exactly one power of a and can be transformed by elementary transformation into one of the eight partial normal forms introduced in Step 1 and 2. By Proposition 2.3.10, one can simply rewrite the powers of a as pairs of powers of a and obtain eight more partial normal forms. Replacing the inverse-free tuple $(h_1, \ldots, h_\mu)$ of short elements by a partial normal form in the resulting tuple of the elementary transformations on $(g_1, \ldots, g_n)$, we finish the proof of the theorem. □

2.3.4 Conjugates of almost short elements and tuples

Suppose that $(g_1, \ldots, g_n)$ is an n-tuple with each g_i conjugate to some almost short element (i.e. the component g_i is conjugate to either a, a^2, b, s_1, t_1 or $ababa$). In this subsection, we first show that by successive application of elementary transformations the n-tuple can be transformed into

$$\prod_{i=1}^{m}(Q_i^{-1}\tau_{i,1}Q_i, \ldots, Q_i^{-1}\tau_{i,n_i}Q_i)$$

with $m \geq 1$, $\sum_{i=1}^{m} n_i = n$, $Q_i \in G$, $\tau_{i,j} \in \mathcal{S}_2$ such that $\tau_{i,1} \cdots \tau_{i,n_i} = 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. For the concatenation of $(g_1, \ldots, g_n)$ and a fixed tuple, we further show a result extremely similar to Theorem 2.1.16.

The first part of this subsection follows a similar line as in Subsection 2.3.2. Proposition 2.3.29 is an analog to Proposition 2.3.10. Lemmata 2.3.22, 2.3.23, 2.3.24, 2.3.25, 2.3.26, 2.3.27 and 2.3.28, which have technicalities referring to Lemma 2.3.9, will be used to prove Proposition 2.3.29.

g_i \ g_{i+1}	a	a^2	b	a^2b	aba	ba^2	ba	a^2ba^2	ab	bab	ba^2b	a^2ba	aba^2	a^2bab	$ababa$	$baba^2$	ba^2ba	$a^2ba^2ba^2$	aba^2b
a	a^2	1	ab	b	a^2ba	aba^2	aba	ba^2	a^2b			ba	a^2ba^2	bab					
a^2	1	a	a^2b	ab	ba	a^2ba^2	a^2ba	aba^2	b			aba	ba^2						ba^2b
b	ba	ba^2	1	ba^2b		a^2	a		bab	ab	a^2b					aba^2	a^2ba		
a^2b	a^2ba	a^2ba^2	a^2			a	1			b	ab					ba^2	aba		
aba	aba^2	ab		a				1	aba^2b			a^2		a^2b				ba^2	
ba^2	b	ba	ba^2b	bab	a			$baba^2$	1				a^2		aba				a^2b
ba	ba^2	b	bab	1	ba^2ba			a^2	ba^2b			a		ab				a^2ba^2	
a^2ba^2	a^2b	a^2ba		a^2bab	1				a^2				a		ba				ab
ab	aba	aba^2	a			1	a^2			a^2b	b					a^2ba^2	ba		
bab			ba			b	ba^2			ba^2b	1						a		
ba^2b			ba^2			ba	b			1	bab					a^2			
a^2ba	a^2ba^2	a^2b		a^2				a				1	$a^2ba^2ba^2$	b				aba^2	
aba^2	ab	aba			a^2				a			$ababa$	1		a^2ba				b
a^2bab			a^2ba			a^2b	a^2ba^2				a^2					$a^2ba^2ba^2$	1		
$ababa$				aba				ab				aba^2		aba^2b				1	
$baba^2$	bab				ba^2				ba				b		ba^2ba				1
ba^2ba		ba^2b		ba^2				ba				b		1				$baba^2$	
$a^2ba^2ba^2$					a^2b				a^2ba^2				a^2ba		1				a^2bab
aba^2b			aba^2			aba	ab			a						1	$ababa$		

TABLE 2.2 – Some pairs (g_i, g_{i+1}) of almost short elements and the products g_ig_{i+1}.

We introduce some pairs of almost short elements in Table 2.2 as in Subsection 2.3.2. Broadly speaking, each pair of almost short elements in Table 2.2 behaves well under the contraction operation introduced in Subsection 2.1.2, which is explained in lemmata 2.3.23, 2.3.24, 2.3.25, 2.3.26, 2.3.27 and 2.3.28. Besides, each pair of almost short elements not in Table 2.2 satisfies the inequality $m_i \leq \min\{\frac{l(g_i)+1}{2}, \frac{l(g_{i+1})+1}{2}\}$ which is the first step for Lemma 2.3.32 (a). (See Lemma 2.3.13 for the precise definition of m_i.) Furthermore, Table 2.2 has to fulfil some irregular requirements which appear in the proofs of Lemma 2.3.32 and Theorem 2.3.33. Unfortunately, we do not have high conviction in sifting out the pairs of almost short elements. What is worse, Theorem 2.3.33 needs a patch based on Lemma 2.3.22 which considers a triple of almost short elements.

Lemma 2.3.22. *Let (τ_1, τ_2, τ_2) be a triple of the form*

$$(a^{-\epsilon}ba^{\epsilon}b, ba^{\epsilon}b, ba^{\epsilon}ba^{-\epsilon}) \text{ or } (ba^{-\epsilon}ba^{\epsilon}, a^{\epsilon}, a^{\epsilon}ba^{-\epsilon}b)$$

with $\epsilon = \pm 1$. Set $(g_1, g_2, g_3) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q, Q^{-1}\tau_3 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2\tau_3 Q \in \mathcal{S}_2$. Then (g_1, g_2, g_3) is Hurwitz equivalent to a triple of almost short elements.

Proof. We only consider the triple $(\tau_1, \tau_2, \tau_2) = (a^{-\epsilon}ba^{\epsilon}b, ba^{\epsilon}b, ba^{\epsilon}ba^{-\epsilon})$.

Since $Q^{-1}\tau_1\tau_2\tau_3 Q = Q^{-1}a^{\epsilon}Q$ is almost short, Q is one of 1, $a^{\pm\epsilon}$, b and $a^{\pm\epsilon}b$. In the case that $Q = b$, the triple $(g_1, g_2, g_3) = (ba^{-\epsilon}ba^{\epsilon}, a^{\epsilon}, a^{\epsilon}ba^{-\epsilon}b)$ is already of almost short elements. In the cases that $Q = a^\epsilon$ or $a^\epsilon b$, the lemma follows from the following substitutions.

$$\begin{aligned}
(a^{-\epsilon}\tau_1 a^\epsilon, a^{-\epsilon}\tau_2 a^\epsilon, a^{-\epsilon}\tau_3 a^\epsilon) &= (a^\epsilon ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon b) \\
&\xrightarrow{R_1} (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon b) \\
&\xrightarrow{R_2} (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon b, ba^\epsilon ba^{-\epsilon}) \xrightarrow{R_1} (a^{-\epsilon}ba^\epsilon b, ba^\epsilon b, ba^\epsilon ba^{-\epsilon}). \\
(ba^{-\epsilon}\tau_1 a^\epsilon b, ba^{-\epsilon}\tau_2 a^\epsilon b, ba^{-\epsilon}\tau_3 a^\epsilon b) &= (ba^\epsilon ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^\epsilon) \\
&\xrightarrow{R_2} (ba^\epsilon ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^\epsilon, a^\epsilon) \\
&\xrightarrow{R_1} (ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon) \xrightarrow{R_2} (ba^{-\epsilon}ba^\epsilon, a^\epsilon, a^\epsilon ba^{-\epsilon}b).
\end{aligned}$$

For the rest two cases, the approach is similar. $\square$

Lemma 2.3.23. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2$ is a power of a. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. The pair (τ_1, τ_2) must be one of

$$\begin{aligned}
&(a^{-\epsilon}, a^{-\epsilon}), \\
&(b, ba^\epsilon), (a^\epsilon b, b), (a^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}), (a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}), (ba^\epsilon, a^{-\epsilon}ba^\epsilon), (a^\epsilon ba^{-\epsilon}, a^\epsilon b), \\
&(a^{-\epsilon}b, ba^{-\epsilon}), (a^\epsilon ba^\epsilon, a^{-\epsilon}b), (ba^{-\epsilon}, a^\epsilon ba^\epsilon), \\
&(a^\epsilon ba^{-\epsilon}b, ba^\epsilon b), (ba^\epsilon b, ba^{-\epsilon}ba^\epsilon)
\end{aligned}$$

and $Q \in \{1, a^\epsilon, a^{-\epsilon}, b, a^\epsilon b, a^{-\epsilon}b\}$ with $\epsilon = \pm 1$. Now we fix $\epsilon = \pm 1$.

When $(\tau_1, \tau_2) = (a^{-\epsilon}, a^{-\epsilon})$, the pair (g_1, g_2) is a pair of almost short elements. When one of g_1, g_2 is conjugate to b and the other one is conjugate to $a^\epsilon b$, either (g_1, g_2) is a pair of almost short elements or (g_1, g_2) is equal to one of $(a^\epsilon b, ba^{-\epsilon}ba^\epsilon b)$, $(ba^\epsilon ba^{-\epsilon}b, ba^\epsilon)$, $(ba^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}b)$ and $(ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}b)$. In this case, the substitutions given by the following graph show that (g_1, g_2) can be transformed into a pair of almost short elements via elementary transformations.

$$\begin{tikzcd}
(b, a^\epsilon b) \arrow[r, "R_1"] & (a^\epsilon b, ba^{-\epsilon}ba^\epsilon b) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}b) \arrow[d, "R_1"] \\
(ba^\epsilon, b) \arrow[u, "R_1"] & (ba^\epsilon ba^{-\epsilon}b, ba^\epsilon) \arrow[l, "R_1"'] & (ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"']
\end{tikzcd}$$

When both of g_1, g_2 are conjugate to $a^{-\epsilon}b$, either (g_1, g_2) is a pair of almost short elements or (g_1, g_2) is one of $(ba^\epsilon ba^\epsilon b, ba^{-\epsilon})$, $(a^{-\epsilon}b, ba^\epsilon ba^\epsilon b)$ which can be transformed into $(ba^{-\epsilon}, a^{-\epsilon}b)$ via R_1, R_1^{-1} respectively. When one of g_1, g_2 is conjugate to a^ϵ and the other one is conjugate to $a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}$, either (g_1, g_2) is a pair of almost short elements or (g_1, g_2) is one of

$$\begin{aligned}
&(ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon), (a^\epsilon ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b), \\
&(a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}), (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}), \\
&(ba^\epsilon ba^\epsilon ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}), (a^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^\epsilon ba^\epsilon b), \\
&(ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^\epsilon ba^{-\epsilon}b), (ba^{-\epsilon}ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b).
\end{aligned}$$

In this case, the following graphs show that (g_1, g_2) can be transformed into a pair of almost short elements via elementary transformations.

$$\begin{tikzcd}
(ba^\epsilon b, ba^{-\epsilon}ba^\epsilon) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon) \arrow[r, "R_1"] & (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^{-\epsilon}b, ba^\epsilon b) \arrow[u, "R_1"] & (a^\epsilon ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"'] & (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}) \arrow[l, "R_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
(ba^\epsilon ba^{-\epsilon}, a^\epsilon) \arrow[d, "R_1"'] & (ba^\epsilon ba^\epsilon ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}) \arrow[l, "R_1"'] & (ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"'] \\
(a^\epsilon, a^{-\epsilon}ba^\epsilon b) \arrow[r, "R_1"'] & (a^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^\epsilon ba^\epsilon b) \arrow[r, "R_1"'] & (ba^{-\epsilon}ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b) \arrow[u, "R_1"']
\end{tikzcd}$$

$\square$

Lemma 2.3.24. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2 = b$. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. The pair (τ_1, τ_2) must be one of

$$(a^\epsilon, a^{-\epsilon}b), (ba^{-\epsilon}, a^\epsilon), (a^{-\epsilon}b, ba^\epsilon b), (ba^\epsilon b, ba^{-\epsilon}), (ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}), (a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon b)$$

with $\epsilon = \pm 1$ and $Q \in \{1, a, a^2, b, ba, ba^2\}$. The lemma follows from the following graphs with $\epsilon = \pm 1$.

$$\begin{tikzcd}
(a^\epsilon, a^\epsilon ba^\epsilon) \arrow[r, "R_1"] & (a^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon) \arrow[d, "R_1"] \\
(a^{-\epsilon}b, a^\epsilon) \arrow[u, "R_1"] & (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}b) \arrow[l, "R_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}) \arrow[r, "R_1"] & (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon) \arrow[u, "R_1"] & (a^\epsilon ba^\epsilon ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^\epsilon) \arrow[l, "R_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
(b, ba^\epsilon ba^{-\epsilon}) \arrow[r, "R_1"] & (ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}ba^\epsilon ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^{-\epsilon}b, b) \arrow[u, "R_1"] & (a^\epsilon ba^{-\epsilon}ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon b) \arrow[r, "R_1"] & (a^{-\epsilon}ba^\epsilon b, a^{-\epsilon}ba^\epsilon b) \arrow[d, "R_1"] \\
(ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon) \arrow[u, "R_1"] & (ba^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^\epsilon) \arrow[l, "R_1"']
\end{tikzcd}$$

$\square$

Lemma 2.3.25. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2 = ba^\epsilon b$ with $\epsilon = \pm 1$. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. The pair (τ_1, τ_2) must be one of

$$(a^\epsilon, a^{-\epsilon}ba^\epsilon b), (ba^\epsilon ba^{-\epsilon}, a^\epsilon), (b, a^\epsilon b), (ba^\epsilon, b), (ba^{-\epsilon}, a^{-\epsilon}b), (ba^{-\epsilon}b, ba^{-\epsilon}b)$$

with $\epsilon = \pm 1$ and $Q \in \{1, b, ba^\epsilon, ba^{-\epsilon}, ba^\epsilon b, ba^{-\epsilon}b\}$. The lemma follows from the following graphs.

$$\begin{tikzcd}
(ba^\epsilon b, ba^{-\epsilon}ba^\epsilon) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon) \arrow[r, "R_1"] & (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^{-\epsilon}b, ba^\epsilon b) \arrow[u, "R_1"] & (a^\epsilon ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"'] & (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}) \arrow[l, "R_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
(a^\epsilon, a^{-\epsilon}ba^\epsilon b) \arrow[r, "R_1"] & (a^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^\epsilon ba^\epsilon b) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b) \arrow[d, "R_1"] \\
(ba^\epsilon ba^{-\epsilon}, a^\epsilon) \arrow[u, "R_1"] & (ba^\epsilon ba^\epsilon ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}) \arrow[l, "R_1"'] & (ba^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
(b, a^\epsilon b) \arrow[r, "R_1"] & (a^\epsilon b, ba^{-\epsilon}ba^\epsilon b) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon b, ba^{-\epsilon}ba^{-\epsilon}b) \arrow[d, "R_1"] \\
(ba^\epsilon, b) \arrow[u, "R_1"] & (ba^\epsilon ba^{-\epsilon}b, ba^\epsilon) \arrow[l, "R_1"'] & (ba^{-\epsilon}ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}b) \arrow[l, "R_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(ba^{-\epsilon}, a^{-\epsilon}b) \arrow[rr, "R_1"] & & (a^{-\epsilon}b, ba^\epsilon ba^\epsilon b) \arrow[dl, "R_1"] \\
& (ba^\epsilon ba^\epsilon b, ba^{-\epsilon}) \arrow[ul, "R_1"] &
\end{tikzcd}$$

$\square$

Lemma 2.3.26. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2 = a^{-\epsilon}ba^\epsilon$ with $\epsilon = \pm 1$. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. The pair (τ_1, τ_2) must be one of

$$(a^\epsilon, a^\epsilon ba^\epsilon), (a^{-\epsilon}, ba^\epsilon), (a^{-\epsilon}ba^{-\epsilon}, a^{-\epsilon}), (a^{-\epsilon}b, a^\epsilon)$$
$$(b, ba^{-\epsilon}ba^\epsilon), (a^{-\epsilon}ba^\epsilon b, b), (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}), (a^\epsilon ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^\epsilon)$$

with $\epsilon = \pm 1$ and $Q \in \{1, a^\epsilon, a^{-\epsilon}, a^{-\epsilon}b, a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}\}$. The lemma follows from the following graphs.

$$\begin{tikzcd}
(a^\epsilon, a^\epsilon ba^\epsilon) \arrow[r, "R_1"] & (a^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^\epsilon) \arrow[d, "R_1"] \\
(a^{-\epsilon}b, a^\epsilon) \arrow[u, "R_1"] & (a^{-\epsilon}ba^\epsilon ba^\epsilon, a^{-\epsilon}b) \arrow[l, "R_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(a^\epsilon, ba^{-\epsilon}) \arrow[r, "R_1"] & (ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^\epsilon, a^\epsilon) \arrow[u, "R_1"] & (a^\epsilon ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^\epsilon) \arrow[l, "R_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
(a^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b) \arrow[r, "R_1"] & (a^\epsilon ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}b) \arrow[d, "R_1"] \\
(ba^\epsilon ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}) \arrow[u, "R_1"] & (ba^\epsilon ba^{-\epsilon}b, ba^\epsilon ba^{-\epsilon}) \arrow[l, "R_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(b, ba^{-\epsilon}ba^\epsilon) \arrow[r, "R_1"] & (ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon ba^{-\epsilon}ba^\epsilon) \arrow[d, "R_1"] \\
(a^{-\epsilon}ba^\epsilon b, b) \arrow[u, "R_1"] & (a^{-\epsilon}ba^\epsilon ba^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^\epsilon b) \arrow[l, "R_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
(a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}) \arrow[r, "R_1"] & (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^{-\epsilon}ba^{-\epsilon}) \arrow[d, "R_1"] \\
(a^\epsilon ba^\epsilon ba^\epsilon, a^{-\epsilon}ba^\epsilon) \arrow[u, "R_1"] & (a^\epsilon ba^\epsilon ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^\epsilon ba^\epsilon) \arrow[l, "R_1"]
\end{tikzcd}$$

□

Lemma 2.3.27. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2$ is one of $a^\epsilon b$, ba^ϵ and $a^{-\epsilon}ba^{-\epsilon}$ with $\epsilon = \pm 1$. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. Since $Q^{-1}\tau_1\tau_2 Q$ is almost short, the element Q is either $(\tau_1\tau_2)^k a^\zeta$ or $(\tau_1\tau_2)^{-l}a^\zeta$ with $k, l \geq 0$ and $\zeta = 0, 1, 2$. When $Q = a^\zeta$, the only exceptional cases that at least one of $Q^{-1}\tau_1 Q$, $Q^{-1}\tau_2 Q$ is not almost short is that (τ_1, τ_2) is equal to one of

$$(b, ba^\epsilon b), (ba^\epsilon b, b), (a^{-\epsilon}b, ba^{-\epsilon}b), (ba^{-\epsilon}b, ba^{-\epsilon})$$

with $\epsilon = \pm 1$, where both $(a^{-\epsilon}\tau_1 a^\epsilon, a^{-\epsilon}\tau_2 a^\epsilon)$ and $(a^\epsilon\tau_1 a^{-\epsilon}, a^\epsilon\tau_2 a^{-\epsilon})$ can be transformed into pairs of almost short elements by applying either R_1 or R_1^{-1}. In general, Lemma 2.3.8 shows that (g_1, g_2) can be transformed into a pair of almost short elements. □

Lemma 2.3.28. *Let (τ_1, τ_2) be a pair of almost short elements in Table 2.2 such that $\tau_1\tau_2$ is almost short and conjugate to ababa. Set $(g_1, g_2) = (Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q)$ with $Q \in G$ and suppose that $Q^{-1}\tau_1\tau_2 Q$ is almost short. Then (g_1, g_2) is Hurwitz equivalent to a pair of almost short elements.*

Proof. Since $\tau_1\tau_2$ is almost short and conjugate to $ababa$, it must be one of $a^{-\epsilon}ba^\epsilon b$, $ba^{-\epsilon}ba^\epsilon$ and $a^\epsilon ba^\epsilon ba^\epsilon$ with $\epsilon = \pm 1$. When $\tau_1\tau_2 = a^{-\epsilon}ba^\epsilon b$, since $Q^{-1}\tau_1\tau_2 Q$ is almost short, the element Q is either $(a^{-\epsilon}ba^\epsilon b)^k a^\zeta$ or $(a^{-\epsilon}ba^\epsilon b)^k(a^\epsilon b)a^\zeta$ with $k \in \mathbb{Z}$ and $\zeta = 0, 1, 2$. Lemma 2.3.8 induces that it suffices to suppose that

$$Q \in \{1, a^\epsilon, a^{-\epsilon}, a^{-\epsilon}b, a^{-\epsilon}ba^\epsilon, a^{-\epsilon}ba^{-\epsilon}\}.$$

Besides, (τ_1, τ_2) is one of

$$(a^{-\epsilon}ba^{-\epsilon}, a^{-\epsilon}b), (a^{-\epsilon}ba^{-\epsilon}ba^{-\epsilon}, a^\epsilon ba^{-\epsilon}b).$$

Each possible (g_1, g_2) is either a pair of almost short elements or transformed into a pair of almost short elements by $R_1^{\pm 1}$. When $\tau_1\tau_2 = ba^{-\epsilon}ba^\epsilon$ or $a^\epsilon ba^\epsilon ba^\epsilon$ we have similar arguments. □

We introduce the following operations and their restorations on an n-tuple $(g_1, \ldots, g_n)$ of elements in G that are conjugate to some almost short element.

- Operation 1 : For $i \in \{1, \ldots, n-1\}$, suppose that $g_i = Q^{-1}\tau_i Q$ and $g_{i+1} = Q^{-1}\tau_{i+1}Q$ with $Q \in G$ and (τ_i, τ_{i+1}) listed in Table 2.2. Then, the operation is a contraction as in Subsection 2.1.2 that replaces (g_i, g_{i+1}) with $g_i g_{i+1}$.
- Operation 1′ : For $i \in \{1, \ldots, n-2\}$, suppose that $g_i = Q^{-1}\tau_i Q$, $g_{i+1} = Q^{-1}\tau_{i+1}Q$ and $g_{i+2} = Q^{-1}\tau_{i+2}Q$ with $Q \in G$ and $(\tau_i, \tau_{i+1}, \tau_{i+2})$ equal to either

  $$(a^2bab, bab, baba^2) \text{ or } (aba^2b, ba^2b, ba^2ba).$$

  Then, the operation is a contraction as in Subsection 2.1.2 that replaces (g_i, g_{i+1}, g_{i+2}) with $g_i g_{i+1} g_{i+2}$.

- Operation 2 : For $i \in \{1, \ldots, n\}$, suppose that $g_i = 1$. The operation moves the identical component to the rightmost position via elementary transformations, removes it and reduces $(g_1, \ldots, g_n)$ to an $(n-1)$-tuple.

Operation 1 and $1'$ are contractions, whose restorations are introduced in Subsection 2.1.1. The restoration of Operation 2 will simply add an identical element on the right side of the tuple.

Proposition 2.3.29. *Let $(g_1, \ldots, g_n)$ be an n-tuple of elements in G which are conjugate to some almost short element such that $g_1 \cdots g_n = 1$. Suppose that we apply the following operations successively on $(g_1, \ldots, g_n)$:*

i) elementary transformations ;
ii) Operation 1 ;
iii) Operation $1'$;
iv) Operation 2 ;

then apply the restorations of Operation 1, $1'$ and 2 in the reverse order. If each component in the resulting tuple before restorations are almost short, then the initial tuple is Hurwitz equivalent to the following tuples :

a) the resulting tuple after restorations ;
b) the concatenation of some tuples of the form $(Q^{-1}\tau_1 Q, Q^{-1}\tau_2 Q, \ldots, Q^{-1}\tau_m Q)$ with $m \geq 1$, $Q \in G$ and $\tau_1, \ldots, \tau_m \in \mathcal{S}_2$ such that $\tau_1 \cdots \tau_m = 1$.

We emphasise that Proposition 2.3.29 does not require an inverse-free tuple $(g_1, \ldots, g_n)$ in G as in Proposition 2.3.10. Besides, an elementary transformation is allowed to transform a pair into such that has a bigger sum of $\mathcal{S}_2$-complexities. That is why we cannot transform it into a tuple of almost short elements but get a concatenation of several tuples of almost short elements each with a diagonal conjugacy.

Proof. Lemma 2.1.8 shows that the initial tuple is Hurwitz equivalent to the resulting tuple after all operations and restorations. We suppose that each component is almost short in the tuple before restorations.

We revisit the introduced operations. Operation 1 may combine $Q^{-1}\tau_i Q$ and $Q^{-1}\tau_j Q$ into $Q^{-1}\tau_i\tau_j Q$ with $Q \in G$ and $\tau_i, \tau_j \in \mathcal{S}_2$. By elementary transformations, the product becomes a conjugate of the form $P^{-1}Q^{-1}\tau_i\tau_j QP$ with some $P \in G$. To restore the operation, we further rewrite it as $(P^{-1}Q^{-1}\tau_i QP, P^{-1}Q^{-1}\tau_j QP)$. Operation $1'$ is similar.

If Operation 2 has never been used, the proposition follows from lemmata 2.3.23, 2.3.24, 2.3.25, 2.3.26, 2.3.27, 2.3.28 and 2.3.22. In general, suppose that $P^{-1}Q^{-1}\tau_i\tau_j QP = 1$. Then $P = 1$ and the restoration replaces the identical element with $(Q^{-1}\tau_i Q, Q^{-1}\tau_j Q)$. We consider the remaining restorations on (τ_i, τ_j) instead. □

Definition 2.3.30. The $\mathcal{S}_2$-*complexity* of an element g conjugate to some element in $\mathcal{S}_2$ is defined as $f_2(g)$ such that

$$f_2(g) = \begin{cases} l(Q) & \begin{array}{l}\text{if } g = Q^{-1}wQ \text{ is almost long} \\ \text{with } Q \in G \text{ and } w \in \{ba^\epsilon b, a^\epsilon ba^{-\epsilon}, a^\epsilon ba^\epsilon, a^\epsilon ba^\epsilon ba^\epsilon | \epsilon = 1, 2\}\,;\end{array} \\ 1/2 & \text{if } g \in \{ababa, a^2ba^2ba^2\}\,; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3.31. Let $(g_1, \ldots, g_n)$ be an n-tuple in G such that each of g_i, $i = 1, \ldots, n$, is conjugate to some element in $\mathcal{S}_2$. A sequence of elementary transformations $(R_{i_1}^{\epsilon_1}, \ldots, R_{i_m}^{\epsilon_m})$, $\epsilon_1, \ldots, \epsilon_m \in \{1, -1\}$, is said to make the sum of $\mathcal{S}_2$-complexities of $(g_1, \ldots, g_n)$ *smaller* if $R_{i_m}^{\epsilon_m} \circ \cdots \circ R_{i_1}^{\epsilon_1}$ transforms $(g_1, \ldots, g_n)$ into a tuple with a smaller sum of $\mathcal{S}_2$-complexities.

Lemma 2.3.32. *Let $(g_1, \ldots, g_n)$ be an n-tuple in G such that each of g_i, $i = 1, \ldots, n$, is conjugate to some element in $\mathcal{S}_2$ and $g_1 \cdots g_n = 1$. Let m_i be the same as in Lemma 2.3.13 and set $m_0 = m_n = 0$ for convenience. Suppose that*

(1) there is no pair of adjacent components g_i, g_{i+1} of the reduced forms $Q^{-1}\tau_i Q$, $Q^{-1}\tau_{i+1} Q$ with $Q \in G$ and (τ_i, τ_{i+1}) in Table 2.2,
(2) there is no sequence of elementary transformations that makes $\sum_i f_2(g_i)$ smaller.

Then m_i, $i = 0, \ldots, n$ have the following properties.

(a) For $i = 1, \ldots, n-1$, $m_i \leq \frac{l(g_i)+1}{2}$ and $m_i \leq \frac{l(g_{i+1})+1}{2}$.
(b) For $i = 1, \ldots, n$, $m_{i-1} + m_i \geq l(g_i)$ only if the reduced form of g_i is either $Q_i^{-1}a^{\epsilon_i}Q_i$ or $Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i$ with $\epsilon_i = 1, 2$, $Q_i \in G$ and $l(Q_i) \geq 0$.
(c) If $m_{i-1} + m_i \leq l(g_i)$ for each of $i = 1, \ldots, n$, then $n = 0$.

Proof. (a) When both g_i and g_{i+1} are almost short, since (g_i, g_{i+1}) does not figure in Table 2.2, we check all possibilities and get that $m_i \leq \frac{l(g_i)+1}{2}, \frac{l(g_{i+1})+1}{2}$.

When $g_i \in \mathcal{S}_2$ but $g_{i+1} \notin \mathcal{S}_2$, we have concluded that $g_{i+1} = Q^{-1}wQ$ with

$$w \in \{ba^\epsilon b, a^\epsilon ba^{-\epsilon}, a^\epsilon ba^\epsilon, a^\epsilon ba^\epsilon ba^\epsilon | \epsilon = 1, 2\}$$

and $Q \in G$ such that $l(Q) \geq 1$. In particular, $l(g_{i+1}) \geq 5 \geq l(g_i)$. Assume that $m_i > \frac{l(g_i)+1}{2}$. Suppose that $l(g_i) = 2$. Then $m_i = 2$ and the symmetry of g_{i+1} implies the contradiction $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$. Suppose that $l(g_i) = 3$. Then $m_i = 3$. The symmetry of g_{i+1} and the fact that $l(w) \geq 3$ imply the contradiction $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$. Suppose that $l(g_i) = 4$. Then $g_i = a^{\epsilon_i} ba^{-\epsilon_i} b$ or $ba^{\epsilon_i} ba^{-\epsilon_i}$ with $\epsilon_i \in \{1, 2\}$. Therefore $m_i = 3$ or 4. If $l(Q) = 1$ and $g_i = a^{\epsilon_i} ba^{-\epsilon} b$, then $m_i = 4$ and $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2). If $l(Q) = 1$ and $g_i = ba^{\epsilon_i} ba^{-\epsilon_i}$, then $g_{i+1} = a^{\epsilon_i} ba^{\epsilon_{i+1}} ba^{-\epsilon_i}$ with $\epsilon_{i+1} \in \{1, 2\}$ and $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2). If $l(Q) \geq 2$, then we again get $l(g_i g_{i+1} g_i^{-1}) < l(g_{i+1})$, contradicting the hypothesis (2). Suppose that $l(g_i) = 5$ and then $m_i = 4$ or 5. As $g_i = a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i}$, m_i must be 5. Therefore, if $l(Q) \geq 2$ then we get the contradiction $l(g_i g_{i+1} g_i^{-1}) < l(g_{i+1})$. If $l(Q) = 1$ then g_{i+1} must be $a^{-\epsilon} ba^{-\epsilon} ba^\epsilon$ but the following substitution makes the sum of $\mathcal{S}_2$-complexities smaller and induces a contradiction.

$$(g_i, g_{i+1}) = (a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i}, a^{-\epsilon} ba^{-\epsilon} ba^\epsilon) \longrightarrow (a^{-\epsilon} ba^{-\epsilon} ba^\epsilon, a^{-\epsilon} ba^\epsilon b) \longrightarrow (a^{-\epsilon} ba^\epsilon b, ba^{-\epsilon} b).$$

We have a similar argument when g_{i+1} is almost short but g_i not.

When both g_i and g_{i+1} are almost long, suppose that their reduced forms are $Q_i^{-1} w_i Q_i$ and $Q_{i+1}^{-1} w_{i+1} Q_{i+1}$ and assume that without loss of generality $l(Q_i) \leq l(Q_{i+1})$. Assume that $m_i > \min\{\frac{l(g_i)+1}{2}, \frac{l(g_{i+1})+1}{2}\}$. Therefore Q_{i+1} must end with Q_i. Write $Q_{i+1} = \tilde{Q} Q_i$ and

$$(g_i, g_{i+1}) = (Q_i^{-1} w_i Q_i, Q_i^{-1} \tilde{Q}^{-1} w_{i+1} \tilde{Q} \tilde{Q}_i).$$

Suppose that $l(\tilde{Q}) = 0$. The assumption on m_i contradicts Table 2.2. Suppose that $l(\tilde{Q}) \geq 1$. Therefore $l(w_i) \leq 5 \leq l(\tilde{Q}^{-1} w_{i+1} \tilde{Q})$ and $m_i > \frac{l(g_i)+1}{2}$. If $l(w_i) = 3$ then $m_i > l(Q_i) + 3$ and $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2). If $l(w_i) = 5$ and $l(\tilde{Q}^{-1} w_{i+1} \tilde{Q}) = 5$, then

$$(g_i, g_{i+1}) = (Q_i^{-1} w_i Q_i, Q_i^{-1} \tilde{Q}^{-1} w_{i+1} \tilde{Q} Q_i) = (Q_i^{-1} a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i} Q_i, Q_i^{-1} a^{-\epsilon_i} ba^{-\epsilon_i} ba^{\epsilon_i} Q_i)$$

whose sum of $\mathcal{S}_2$-complexities can be smaller using elementary transformations. If $l(w_i) = 5$ and $l(\tilde{Q}^{-1} w_{i+1} \tilde{Q}) \geq 7$, then $l(g_i g_{i+1} g_i^{-1}) \leq l(g_{i+1}) - 2$, contradicting the hypothesis (2).

(b) Suppose that $m_{i-1} + m_i \geq l(g_i)$ for some $i = 1, \ldots, n-1$.

Suppose that $g_i \in \mathcal{S}_2$. If $g_i = b$ then either $m_{i-1} = 0$, $m_i = 1$ or $m_{i-1} = 1$, $m_i = 0$. Therefore either g_{i-1} ends with b or g_{i+1} starts with b. Table 2.2 shows that either g_{i-1} or g_{i+1} is almost long, starts and ends with b. Hence it implies the contradiction either $l(g_i g_{i+1} g_i^{-1}) < l(g_{i+1})$ or $l(g_i^{-1} g_{i-1} g_i) < l(g_{i-1})$. If $l(g_i) = 2$, then one of g_{i-1}, g_{i+1} must be b, which is impossible based on Table 2.2. If $l(g_i) = 4$, then either $g_{i-1} = a^{\epsilon_{i-1}} b$ or $g_{i+1} = ba^{\epsilon_{i+1}}$ with $\epsilon_{i-1}, \epsilon_{i+1} \in \{1, 2\}$, which is impossible based on Table 2.2. If $l(g_i) = 3$ and $g_i = a^{\epsilon_i} ba^{\epsilon_i}$ with $\epsilon_i \in \{1, 2\}$, then either $g_{i-1} = ba^{-\epsilon_i}$ or $g_{i+1} = a^{-\epsilon_i} b$. If $l(g_i) = 3$ and $g_i = a^{\epsilon_i} ba^{-\epsilon_i}$ with $\epsilon_i \in \{1, 2\}$, then either $g_{i-1} = ba^{-\epsilon_i}$ or $g_{i+1} = a^{\epsilon_i} b$. Both are impossible again based on Table 2.2. There are only two possibilities left : either g_i is conjugate to a power of a or $g_i = a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i}$.

When g_i is almost long, g_i is one of

$$Q^{-1} ba^{\epsilon_i} bQ, Q^{-1} a^{\epsilon_i} ba^{-\epsilon_i} Q, Q^{-1} a^{\epsilon_i} ba^{\epsilon_i} Q, Q^{-1} a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i} Q$$

with $\epsilon_i \in \{1, 2\}$, $Q \in G$ and $l(Q) \geq 1$. If $g_i = Q^{-1} a^{\epsilon_i} ba^{\epsilon_i} Q$ or $g_i = Q^{-1} a^{\epsilon_i} ba^{-\epsilon_i} Q$ then either $g_{i+1} = Q^{-1} a^{-\epsilon_i} b$ or $g_{i+1} = Q^{-1} a^{\epsilon_i} b$. Therefore $g_{i+1}^{-1} g_i g_{i+1} \in \{Q^{-1} ba^{2\epsilon_i} Q, Q^{-1} bQ\}$ which is a contradiction.

We conclude that g_i is either $Q_i^{-1} a^{\epsilon_i} Q_i$ or $Q_i^{-1} a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i} Q_i$ with $\epsilon_i = 1, 2$ and $l(Q_i) \geq 0$.

(c) We assume that $n \geq 1$ and suppose that $m_{i-1} + m_i = l(g_i)$ for some $2 \leq i \leq n-1$. By (2), g_i is either $Q_i^{-1} a^{\epsilon_i} Q_i$ or $Q_i^{-1} a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i} Q_i$ with $\epsilon_i = 1, 2$ and $l(Q_i) \geq 0$.

If $g_i = a^{\epsilon_i}$ and suppose that $m_{i-1} = 0$, $m_i = 1$, then g_{i+1} is either one of $a^{\epsilon_i} ba^{\epsilon_i} ba^{\epsilon_i}$, $a^{\epsilon_i} ba^{-\epsilon_i} b$, $a^{-\epsilon_i} ba^{-\epsilon_i} ba^{-\epsilon_i}$ or an almost long element starting with a^{ϵ_i} and ending with $a^{-\epsilon_i}$. However, g_{i+1} cannot be $a^{-\epsilon_i} ba^{-\epsilon_i} ba^{-\epsilon_i}$ since the elementary transformation R_i^{-1} makes the sum of $\mathcal{S}_2$-complexities smaller.

If $g_i = Q_i^{-1} a^{\epsilon_i} Q_i$ with $l(Q_i) \geq 1$, then either $g_{i-1} = Q_i$ or $g_{i+1} = Q_i^{-1}$, which implies the contradiction either $g_{i-1} g_i g_{i-1}^{-1} = a^{\epsilon_i}$ or $g_{i+1}^{-1} g_i g_{i+1} = a^{\epsilon_i}$.

If $g_i = Q^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q$ with $l(Q) \geq 0$, then either $g_{i-1} = ba^{-\epsilon_i}Q$ or $g_{i+1} = Q^{-1}a^{-\epsilon_i}b$. Therefore Table 2.2 denies the case of $Q = 1$ and, when $Q \neq 1$, either $g_{i-1}g_ig_{i-1}^{-1} = a^{\epsilon_i}ba^{2\epsilon_i}b$ or $g_{i+1}^{-1}g_ig_{i+1} = ba^{2\epsilon_i}ba^{\epsilon_i}$, which is a contradiction.

The assertion (b) and the above observation show that $m_{i-1} + m_i < l(g_i)$ if $g_i \neq a^{\epsilon_i}$. They further imply a contradiction that $g_1 \cdots g_n \neq 1$. □

Now we state the main result in this subsection.

Theorem 2.3.33. *Let $g_1, \ldots, g_n$ be such that each of them is conjugate to some element in $\mathcal{S}_2$ and $g_1 \cdots g_n = 1$. Then, the n-tuple $(g_1, \ldots, g_n)$ is Hurwitz equivalent to*

$$\prod_{i=1}^{m}(Q_i^{-1}\tau_{i,1}Q_i, \ldots, Q_i^{-1}\tau_{i,n_i}Q_i)$$

with $m \geq 1$, $\sum_{i=1}^{m} n_i = n$, $Q_i \in G$ and $\tau_{i,j} \in \mathcal{S}_2$ such that $\tau_{i,1} \cdots \tau_{i,n_i} = 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$.

Proof. We will always use the notation m_i to indicate the length of the reduced part in h_ih_{i+1} for $i = 1, \ldots, \mu - 1$ and set $m_0 = m_\mu = 0$ as before. To prove the theorem for $(g_1, \ldots, g_n)$, we use the induction on

$$\Big(n, \sum_{i=1}^{n} f_2(g_i), l(g_1), \ldots, l(g_n)\Big).$$

and apply the following operations : If there exists a pair of adjacent components of the form $(Q^{-1}\tau_1Q, Q^{-1}\tau_2Q)$ with $Q \in G$ and (τ_1, τ_2) in Table 2.2, then we replace it with the product $Q^{-1}\tau_1\tau_2Q$ and reduce $(g_1, \ldots, g_n)$ to an $(n-1)$-tuple. If there exists a triple of consecutive components of the form

$$(Q^{-1}a^{-\epsilon}ba^{\epsilon}bQ, Q^{-1}ba^{\epsilon}bQ, Q^{-1}ba^{\epsilon}ba^{-\epsilon}Q)$$

with $Q \in G$, $\epsilon = \pm 1$ as introduced in Operation 1$'$, then we replace it with $Q^{-1}a^{\epsilon}Q$ and reduce $(g_1, \ldots, g_n)$ to an $(n-2)$-tuple. If there exists an identical component, then we move it to the rightmost position and remove it. If there exists a sequence of elementary transformations that makes $\sum_i f_2(h_i)$ smaller, then we apply it.

When each of the above operations fails, the resulting tuple, still denoted by $(g_1, \ldots, g_n)$, satisfies all hypotheses in Lemma 2.3.32. Suppose that $n \geq 1$ and there exists some $i = 2, \ldots, n-1$ such that $m_{i-1} + m_i > l(g_i)$.

When $g_i = a^{\epsilon_i}$ with $\epsilon_i = 1, 2$, Table 2.2 reveals that either g_{i-1} is one of $ba^{-\epsilon_i}ba^{\epsilon_i}$, $a^{-\epsilon_i}ba^{-\epsilon_i}ba^{-\epsilon_i}$, $a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}$, or g_{i-1} is an almost long element starting with $a^{-\epsilon_i}$ and ending with a^{ϵ_i}. Meanwhile, either g_{i+1} is one of $a^{\epsilon_i}ba^{-\epsilon_i}b$, $a^{-\epsilon_i}ba^{-\epsilon_i}ba^{-\epsilon_i}$, $a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}$, or g_{i+1} is an almost long element starting with a^{ϵ_i} and ending with $a^{-\epsilon_i}$. The triple (g_{i-1}, g_i, g_{i+1}) cannot be $(ba^{-\epsilon_i}ba^{\epsilon_i}, a^{\epsilon_i}, a^{\epsilon_i}ba^{-\epsilon_i}b)$ due to Operation 1$'$. Therefore, either (g_{i-1}, g_i) can be transformed into $(\tilde{g}_{i-1}, \tilde{g}_i) = (a^{\epsilon_i}, a^{-\epsilon_i}g_{i-1}a^{\epsilon})$ with $f_2(g_{i-1}) \geq f_2(\tilde{g}_i)$ but $l(\tilde{g}_{i-1}) < l(g_{i-1})$.

When $g_i = ba^{\epsilon_i}b$ with $\epsilon_i = 1, 2$, Table 2.2 reveals that either $g_{i-1} = a^{-\epsilon_i}ba^{\epsilon_i}b$ or g_{i-1} is almost long starting with $ba^{-\epsilon_i}$ and ending with $a^{\epsilon_i}b$. Meanwhile, either $g_{i+1} = ba^{\epsilon}ba^{-\epsilon_i}$ or g_{i+1} is almost long starting with ba^{ϵ} and ending with $a^{-\epsilon}b$. The triple (g_{i-1}, g_i, g_{i+1}) cannot be $(a^{-\epsilon_i}ba^{\epsilon_i}b, ba^{\epsilon_i}b, ba^{\epsilon}ba^{-\epsilon_i})$ due to Operation 1$'$. Therefore, (g_{i-1}, g_i, g_{i+1}) can be transformed into $(\tilde{g}_{i-1}, \tilde{g}_i, \tilde{g}_{i+1})$ with $f_2(g_{i-1}) + f_2(g_i) + f_2(g_{i+1}) \geq f_2(\tilde{g}_{i-1}) + f_2(\tilde{g}_i) + f_2(\tilde{g}_{i+1})$, $l(g_{i-1}) \geq l(\tilde{g}_{i-1})$, $l(g_i) \geq l(\tilde{g}_i)$, $l(g_{i+1}) \geq l(\tilde{g}_{i+1})$ but either $l(\tilde{g}_{i-1}) < l(g_{i-1})$ or $l(\tilde{g}_{i+1}) < l(g_{i+1})$.

When $g_i = Q_i^{-1}a^{\epsilon_i}Q$ with $\epsilon_i = 1, 2$, $l(Q_i) \geq 2$, we have $m_{i-1} = m_i = l(Q_i) + 1$ and $l(g_{i-1}) > l(g_i)$. To avoid $l(g_i^{-1}g_{i-1}g_i) < l(g_{i-1})$, g_{i-1} must end with $a^{\epsilon_i}Q_i$ and start with $Q_i^{-1}a^{-\epsilon_i}$. In this case, $l(g_i^{-1}g_{i-1}g_i) = l(g_{i-1})$ and, using the elementary transformation R_{i-1}, we are able to reduce $(g_1, \ldots, g_n)$ to a new n-tuple, say $(\tilde{g}_1, \ldots, \tilde{g}_n)$, such that $\sum_j f_2(g_j) = \sum_j f_2(\tilde{g}_j)$, $l(g_j) = l(\tilde{g}_j)$ for $1 \leq j \leq n$ and $j \notin \{i-1, i\}$ but $l(\tilde{g}_{i-1}) = l(g_i) < l(g_{i-1}) = l(\tilde{g}_i)$.

When $g_i = Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i$ with $\epsilon_i = 1, 2$, $Q_i \in G$ and $l(Q_i) \geq 0$, then $m_{i-1} = m_i = l(Q_i) + 3$. If $l(g_{i-1}) = l(g_i) = l(g_{i+1})$, then

$$(g_{i-1}, g_i, g_{i+1}) = (Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{-\epsilon_i}Q_i, Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i)$$

that can be transformed into a triple with a smaller sum of $\mathcal{S}_2$-complexities via the following substitution.

$$\begin{aligned}(g_{i-1}, g_i, g_{i+1}) &\longrightarrow (Q_i^{-1}a^{\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i)\\ &\longrightarrow (Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{-\epsilon_i}ba^{-\epsilon_i}ba^{-\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i)\\ &\longrightarrow (Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}ba^{\epsilon_i}Q_i, Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}bQ_i).\end{aligned}$$

If $l(g_{i-1}) = l(g_i) < l(g_{i+1})$, then $g_{i+1} = Q_i^{-1}a^{-\epsilon_i}ba^{\epsilon_i}wa^{-\epsilon_i}ba^{\epsilon_i}Q_i$ with the word w starts and ends with b. Therefore, by elementary transformations the triple can be transformed into

$$(Q_i^{-1}a^{\epsilon_i}bwba^{-\epsilon_i}Q, g_{i-1}, g_i)$$

with a smaller sum of $\mathcal{S}_2$-complexities, which induces a contradiction. If $l(g_{i-1}) > l(g_i)$ then again using the elementary transformation R_{i-1} we are able to reduce $(g_1, \ldots, g_n)$ to a new n-tuple, say $(\tilde{g}_1, \ldots, \tilde{g}_n)$, such that $\sum_j f_2(g_j) = \sum_j f_2(\tilde{g}_j)$, $l(g_j) = l(\tilde{g}_j)$ for $1 \leq j \leq n$ and $j \notin \{i-1, i\}$ but $l(\tilde{g}_{i-1}) = l(g_i) < l(g_{i-1}) = l(\tilde{g}_i)$.

The induction does not stop unless n is equal to 0. Due to Proposition 2.3.29, by restoring operations and applying more elementary transformations, we get a resulting n-tuple of almost short elements that can be obtained from the original $(g_1, \ldots, g_n)$ via elementary transformations directly. □

Corollary 2.3.34. *Let $g_1, \ldots, g_n$ be such that each of them is conjugate to some element in $\mathcal{S}_2$ and $g_1 \cdots g_n = 1$. Let $(g'_1, \ldots, g'_m)$ be a tuple containing a generating set. Then, the tuple $(g_1, \ldots, g_n) \bullet (g'_1, \ldots, g'_m)$ is Hurwitz equivalent to $(h_1, \ldots, h_n) \bullet (g'_1, \ldots, g'_m)$ where $(h_1, \ldots, h_n)$ is an n-tuple of almost short elements.*

Proof. The corollary follows from Theorem 2.3.33 and Lemma 2.1.6. □

Recall that $\mathcal{F}_{13} = (b, b)^2 \bullet (a^2bab, t_0, s_1)^3$. Theorem 2.1.17 and Theorem 2.1.18 show that we are able to construct the normal form of $(g_1, \ldots, g_n) \bullet \mathcal{F}_{13}$ that depends only on the number of components in each conjugacy class as in Theorem 2.1.16. We prove them using the following.

Theorem 2.3.35. *Let $g_1, \ldots, g_n \in \mathrm{PSL}(2, \mathbb{Z})$ be conjugates of a, a^2, b, aba, a^2ba^2 or $ababa$ satisfying $g_1 \cdots g_n = 1$. Suppose that m of them are conjugates of $ababa$. Let $(v_1, \ldots, v_c)$ be a tuple containing a generating set, and let $(v'_1, \ldots, v'_{c'})$ be a (b, b, b, b)-expanding tuple whose components are conjugate to a, a^2, b, s_0 or t_0. Then,*

$$(g_1, \ldots, g_n) \bullet (v'_1, \ldots, v'_{c'}) \bullet (v_1, \ldots, v_c)$$

is Hurwitz equivalent to

$$(h_1, \ldots, h_{n'}) \bullet (a^2bab, ba^2ba)^{(m-3+\mu)/2} \bullet (v_1, \ldots, v_c)$$

where all components of $(h_1, \ldots, h_{n'})$ are conjugate to a, a^2, b, s_0, t_0, $ababa$ and only $3 - \mu$ of them are conjugate to $ababa$, where $\mu = 3 - m$ if $m \leq 3$ and $\mu = (m + 1) \mod 2$ otherwise.

Proof. We assume that $(v'_1, \ldots, v'_{c'}) = (b, b, b, b)$ without loss of generality. Rewrite $(g_1, \ldots, g_n) \bullet (b, b, b, b) \bullet (v_1, \ldots, v_c)$ as

$$(h_1, \ldots, h_k) \bullet (b, b) \bullet (v_1, \ldots, v_c) \bullet (a^2bab, ba^2ba)^l,$$

with $(h_1, \ldots, h_k) = (g_1, \ldots, g_n, b, b)$ containing at least one component conjugate to b, with $k = n + 2$ and $l = 0$. Following Corollary 2.3.34, we transform $(h_1, \ldots, h_k)$ into a tuple of almost short elements that contains at least one component of the form either b, a^2ba or aba^2.

We first apply the following inductions on k when $m - 2l > 3$.

Suppose that there exist two components, say h_i and h_j with $i \neq j$, such that both of them are conjugate to $ababa$ and $h_ih_j = 1$. We move h_i and h_j to the rightmost positions. Since $(v_1, \ldots, v_c)$ contains a generating set, by Lemma 2.1.6, they are further transformed into a pair of the form (a^2bab, ba^2ba) by elementary transformations on $(g_1, \ldots, g_n) \bullet (b, b) \bullet (v_1, \ldots, v_c)$. Therefore, we get the following tuple

$$(\tilde{h}_1, \ldots, \tilde{h}_{k-2}) \bullet (b, b) \bullet (v_1, \ldots, v_c) \bullet (a^2bab, ba^2ba)^{l+1}$$

where $(\tilde{h}_1, \ldots, \tilde{h}_{k-2})$ is further transformed into a tuple of almost short elements.

Suppose that with a pair

$$(\tau_b, \tau_{ababa}) \in \left\{ \begin{array}{c} (b, a^2bab), (b, ba^2ba), (b, baba^2), (b, aba^2b), \\ (aba^2, ababa), (aba^2, a^2ba^2ba^2), (aba^2, baba^2), (aba^2, aba^2b), \\ (a^2ba, ababa), (a^2ba, a^2ba^2ba^2), (a^2ba, a^2bab), (a^2ba, ba^2ba) \end{array} \right\}$$

there exist some components of the form τ_b and at least two components of the form τ_{ababa}. Using elementary transformations we gather them together and obtain a pair of mutually inverse elements via the following substitution.

$$(\tau_b, \tau_{ababa}, \tau_{ababa}) \longrightarrow (\tau_{ababa}^{-1}, b, \tau_{ababa}) \longrightarrow (\tau_{ababa}^{-1}, \tau_{ababa}, \tau_{ababa}^{-1}b\tau_{ababa}).$$

The pair of mutually inverse elements is further moved to the rightmost position and transformed into (a^2bab, ba^2ba). The resulting tuple again has the expression with a lower k.

Once the above induction stops but $m-2l>3$, there is at most one almost short element conjugate to $ababa$, say τ_{abab}, that appears more than once in $(h_1,\dots,h_k)$. Take a proper $\tau_b \in \{b, a^2ba, aba^2\}$ such that (τ_b, τ_{ababa}) belongs to the above set of pairs. Transform the extra pair (b,b) into (τ_b,τ_b) with the help of $(v_1,\dots,v_c)$. Again we gather all components of the form τ_{ababa} in $(h_1,\dots,h_k)$ together with an additional τ_b using elementary transformations and apply the following substitutions.

$$(\tau_b, \tau_{ababa}, \dots, \tau_{ababa}) \longrightarrow (\tau_{ababa}^{-1}, b, \tau_{ababa}, \dots, \tau_{ababa}) \longrightarrow (\tau_{ababa}^{-1}, \dots, \tau_{ababa}^{-1}, b, \tau_{ababa}, \dots, \tau_{ababa}).$$

We make all mutually inverse elements within the above resulting tuple pairs of the form (a^2bab, ba^2ba) and move them to the rightmost positions. By elementary transformations the tuple $(g_1,\dots,g_n)\bullet(v'_1,\dots,v'_{c'})\bullet(v_1,\dots,v_c)$ has been finally transformed into

$$(h_1,\dots,h_k)\bullet(v_1,\dots,v_c)\bullet(a^2bab, ba^2ba)^{m'/2}$$

where $(h_1,\dots,h_k)$ is a tuple of almost short elements containing at most three components conjugate to $ababa$ and $m-m'\le 3$. □

Proof of Theorem 2.1.17 and Theorem 2.1.18. We only prove Theorem 2.1.17 while the proof of Theorem 2.1.18 is similar. Since (a^2bab, t_0, s_1) contains a generating set, Theorem 2.3.35 has shown that the concatenation $(g_1,\dots,g_n)\bullet\mathcal{F}_{13}$ can be transformed into

$$(h_1,\dots,h_{n'})\bullet(a^2bab, ba^2ba)^{(m-3+\mu)/2}\bullet(a^2bab, t_0, s_1)^3$$

where μ is determined by m, only $3-\mu$ components of $(h_1,\dots,h_{n'})$ are conjugate to $ababa$ and the rest are conjugates of a, a^2, b, s_0 or t_0.

Consider each of $i=1,2,3$ in turn. Let h_α be the first component in $(h_1,\dots,h_{n'})$ conjugate to $ababa$. Since the first triple of the form (a^2bab, t_0, s_1) within the concatenation contains a genearting set, by Lemma 2.1.6 we can transform $(h_1,\dots,h_{n'})$ into a tuple with a simultaneous conjugation such that $h_\alpha = ba^2ba$ Therefore we are able to make the α-th component in $(h_1,\dots,h_{n'})$ and the first component in the triple a pair of the form (a^2bab, ba^2ba). Hence $(h_1,\dots,h_{n'})\bullet(a^2bab,t_0,s_1)^{4-i}$ is transformed into $(h'_1,\dots,h'_{n''})\bullet(a^2bab,t_0,s_1)^{3-i}\bullet(a^2bab,ba^2ba)$ with $n''=n'+1$. □

2.4 Stable vs. unstable classifications of torus fibrations

2.4.1 Hurwitz equivalence fails without stabilisation

We give some examples of global monodromies which are Hurwitz equivalent up to stabilisation, as in Theorem B and Theorem C, but fail to be Hurwitz equivalent. This illustrates why it is necessary to consider fibrations up to the fibre-connected sum.

Example 2.4.1. Let f_1 be an achiral Lefschetz fibration which has a global monodromy of the form $(-A^2B, -BA, -A^2B, -BA)$ and let f_2 be an achiral Lefschetz fibration which has a global monodromy of the form $(-A^2B, -BA, -ABA, A^2BA^2)$. Though f_1 and f_2 have the same type of singularities, these two global monodromies are not Hurwitz equivalent.

Proof. Indeed, the following graph shows all resulting tuples in $\mathrm{PSL}(2,\mathbb{Z})$ from (s_0,t_0,s_0,t_0) using elementary transformations.

$$\begin{tikzcd}
& & (s_0,t_0,t_0,s_0) \arrow[loop below, "R_2^{\pm 1}"] & & \\
(s_0,t_0,s_0,t_0) \arrow[urr, leftrightarrow, bend left, "R_3^{\pm 1}"] \arrow[r, leftrightarrow, "R_2^{\pm 1}"] \arrow[drr, leftrightarrow, bend right, "R_1^{\pm 1}"'] & (s_0,s_0,t_0,t_0) \arrow[loop above, "R_1^{\pm 1}"] \arrow[loop below, "R_3^{\pm 1}"] & & (t_0,t_0,s_0,s_0) \arrow[loop above, "R_1^{\pm 1}"] \arrow[loop below, "R_3^{\pm 1}"] \arrow[r, leftrightarrow, "R_2^{\pm 1}"] & (t_0,s_0,t_0,s_0) \arrow[ull, leftrightarrow, bend right, "R_1^{\pm 1}"'] \arrow[dll, leftrightarrow, bend left, "R_3^{\pm 1}"] \\
& & (t_0,s_0,s_0,t_0) \arrow[loop above, "R_2^{\pm 1}"] & &
\end{tikzcd}$$

In particular, one cannot transform (s_0,t_0,s_0,t_0) into (s_0,t_0,s_1,t_1). □

Example 2.4.2. Let (b, b, a^2bab, ba^2ba) and $(aba^2, a^2ba, a^2bab, baba^2)$ be tuples in $\mathrm{PSL}(2,\mathbb{Z})$. We claim that for arbitrary positive integer N,

$$(b, b, a^2bab, ba^2ba) \bullet (b, b)^N$$

cannot be transformed into

$$(aba^2, a^2ba, a^2bab, baba^2) \bullet (b, b)^N$$

by elementary transformations.

Proof. Assume that $(b, b, a^2bab, ba^2ba) \bullet (b, b)^N$ can be transformed into $(aba^2, a^2ba, a^2bab, baba^2) \bullet (b, b)^N$ by elementary transformations for some N. Then there exists an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ which is a product of b, a^2bab and ba^2ba such that $aba^2 = g^{-1}bg$, which implies $b = (ga)^{-1}b(ga)$. Therefore, the element g is either a^2 or ba^2, but the number of occurrences of the letter a in g modulo 3 is equal to 0, which is a contradiction. □

2.4.2 Unstable classification of achiral Lefschetz fibrations

We consider torus achiral Lefschetz fibrations having fixed cardinality of branch sets $|\mathcal{B}| = n \geq 1$. A singular fibre is of type I_1^+ if its fibre monodromy is conjugate to $L = -ABA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and a singular fibre is of type I_1^- if its fibre monodromy is conjugate to $R = -AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

By Theorem B, global monodromies of a pair of torus achiral Lefschetz fibrations are Hurwitz equivalent after performing direct sums with f_{12}^L if and only if they have the same type of singularities. However, the Hurwitz equivalence between global monodromies is more difficult to state, especially when singular fibres of type I_1^+ and I_1^- occur in pairs. In this subsection, we enumerate all possible Hurwitz equivalent classes of global monodromies of torus achiral Lefschetz fibrations, without stabilisation. This will prove Theorem D.

By a *rooted tree* we mean a directed tree in which a specific vertex is called the *root*, such that each directed edge indicates the parent-child relationship between two vertices. A *rooted forest* is a disjoint union of several rooted trees. In general, given a (directed) graph Γ, we always use $V(\Gamma)$ to denote the set of vertices.

Definition 2.4.3. Given a rooted forest T and a non-negative integer k, we define $\Omega(T, k)$ to be the set of formal sums $\sum_{v \in V(T)} m_v \cdot v$ over vertices with $m_v \geq 0$ and $\sum m_v = k$ such that any two vertices $v_1 \neq v_2$ with $m_{v_1} \geq 1$, $m_{v_2} \geq 1$ have no ancestor-descendant relationship (i.e. there does not exist a directed path joining v_1 to v_2).

Definition 2.4.4. For $n = p + q$ with $p \geq 0$, $q \geq 0$, we define $\mathrm{Hom}_{p,q}^{\mathrm{aL}}(\mathbb{F}_{n-1}, \mathrm{SL}(2,\mathbb{Z}))$ to be the set consisting of all monodromy homomorphisms of torus achiral Lefschetz fibrations $f : M \to S^2$ with $\mathcal{O}(f) = [\underbrace{I_1^+, \ldots, I_1^+}_{p \text{ components}}, \underbrace{I_1^-, \ldots, I_1^-}_{q \text{ components}}]$.

Theorem 2.4.5. *Let n, p and q be arbitrary integers such that $n \geq 1$, $p \geq 0$, $q \geq 0$ and $p + q = n$.*
- *If $p \neq q$, then the set $B_n \backslash \mathrm{Hom}_{p,q}^{aL}(\mathbb{F}_{n-1}, \mathrm{SL}(2,\mathbb{Z}))$ is a singleton.*
- *If $p = q$, then there exists a one-to-one correspondence :*

$$B_n \backslash \mathrm{Hom}_{p,q}^{aL}(\mathbb{F}_{n-1}, \mathrm{SL}(2,\mathbb{Z})) \longleftrightarrow$$
$$\{pt.\} \sqcup \Big(\bigsqcup_{k=0}^{p-1} \Omega(T_\infty, k) \Big) \sqcup \Big(\bigsqcup_{k=0}^{p-1} \Omega(T_\infty, k) \Big) \sqcup \Big(\bigsqcup_{k=0}^{p-1} \Omega(T_\infty, k) \Big) \sqcup \Omega(T_\infty \sqcup T_\infty \sqcup T_\infty, p)$$

where T_∞ is the rooted complete infinite binary tree.

Recall that each matrix g with non-zero trace in $\mathrm{SL}(2,\mathbb{Z})$ is uniquely expressed by ϵQ with $\epsilon = \pm I$ and Q a word in $\{A, A^2, B\}$ in which B's and powers of A appear alternatively. The *length* of an element $g \in \mathrm{SL}(2,\mathbb{Z})$ is defined as the length of the word Q, denoted by $l(g)$. Here we list all possibilities for fibre monodromies of a torus achiral Lefschetz fibration :

$$-A^2B, -ABA, -BA^2 \text{ and } \epsilon PABAQ,$$
$$-BA, A^2BA^2, -AB \text{ and } \epsilon PA^2BA^2Q$$

where P, Q are words in $\{A, A^2, B\}$ in which B's and powers of A appear alternatively, $PQ = \pm I$, $\epsilon = \pm I$ is uniquely determined by $l(Q)$ such that the trace is equal to $+2$.

Let g_1 and g_2 be matrices in $\mathrm{SL}(2,\mathbb{Z})$. Suppose that g_1 and g_2 are expressed by $\epsilon t_k \dots t_1$ and $\tilde{\epsilon}\tilde{t}_1 \dots \tilde{t}_l$, respectively, with $\epsilon, \tilde{\epsilon} \in \{I, -I\}$, $k = l(g_1)$, $l = l(g_2)$, $t_j \in \{A, A^2, B\}$, $j = 1, \dots, k$ and $\tilde{t}_j \in \{A, A^2, B\}$, $j = 1, \dots, l$. The product $g_1 g_2$ is either

$$\tau t_k \dots t_m r \tilde{t}_{m+1} \dots \tilde{t}_l \text{ or } \tau t_k \dots t_m r \text{ or } \tau r \tilde{t}_{m+1} \dots \tilde{t}_l$$

for some $\tau = \tau(g_1, g_2) \in \{I, -I\}$, $r = r(g_1, g_2) \in G$ such that $l(r) \leq 1$ and $0 \leq m = m(g_1, g_2) \leq k, l$.

Let $Q = BA^{k_1}BA^{k_2} \cdots BA^{k_m}B^{\lambda}$ be a matrix in $\mathrm{SL}(2,\mathbb{Z})$ with $m \geq 0$, $k_1, \dots, k_m \in \{1, 2\}$ and $\lambda \in \{0, 1\}$. We introduce the *suffix tree* T_Q which is a rooted binary tree with infinitely many vertices, whose each vertex is labelled with a pair of inverse elements in $\mathrm{SL}(2,\mathbb{Z})$. Set

$$\widetilde{Q} = B^{\lambda}A^{3-k_m}B \cdots A^{3-k_2}BA^{3-k_1}B$$

so that $\widetilde{Q}Q = \pm I$ and let $\epsilon = \pm I$ be such that $\mathrm{trace}(\epsilon\widetilde{Q}ABAQ) = 2$. The root of T_Q is labelled with the pair

$$(\epsilon\widetilde{Q}ABAQ, -\epsilon\widetilde{Q}A^2BA^2Q)$$

and the suffix tree T_Q is defined by the following form iteratively, where each directed edge indicates the parent-child relationship between a vertex and the root of a suffix tree.

$$\begin{tikzcd}
& (\epsilon\widetilde{Q}ABAQ, -\epsilon\widetilde{Q}A^2BA^2Q) \arrow[dl] \arrow[dr] & \\
T_{BAQ} & & T_{BA^2Q}
\end{tikzcd}$$

All conjugates of L and R occur in pairs. They are in one-to-one correspondence with the vertices in the following infinite directed graph Γ, where each directed edge again indicates the parent-child relationship between a vertex and the root of a suffix tree.

$$\begin{tikzcd}
(-A^2B, -BA) \arrow[dr] \arrow[drr] & (-ABA, A^2BA^2) \arrow[dl] \arrow[dr] & (-BA^2, -AB) \arrow[dll] \arrow[dl] \\
T_{BA^2} & T_B & T_{BA}
\end{tikzcd}$$

The vertices labelled by $(-A^2B, -BA)$, $(-ABA, A^2BA^2)$ or $(-BA^2, -AB)$ are called *exceptional.* The components of these labels project to short elements in $\mathrm{PSL}(2,\mathbb{Z})$, as in Section 2.3.

Let $(g_1, \dots, g_n)$ be a global monodromy of torus achiral Lefschetz fibrations, which is an n-tuple of elements conjugate to either L or R. The *complexity* of this tuple is defined to be

$$\mathrm{cxty}(g_1, \dots, g_n) := \sum_i \mathrm{cxty}(g_i), \qquad \mathrm{cxty}(g_i) = \begin{cases} l(Q) & \text{if } g_i = \epsilon\widetilde{Q}ABAQ \text{ or } g_i = \epsilon\widetilde{Q}A^2BA^2Q \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 2.4.6. *Let (e_1, e_1^{-1}), $(e_2, e_2^{-1}) \in V(\Gamma)$ be distinct vertices.*

(a) If there exists an ancestor-descendant relationship between (e_1, e_1^{-1}) and (e_2, e_2^{-1}), then by a sequence of elementary transformations the quadruple

$$(e_1, e_1^{-1}, e_2, e_2^{-1})$$

can be transformed into a quadruple of the form

$$(e'_1, {e'}_1^{-1}, e'_2, {e'}_2^{-1})$$

such that $\mathrm{cxty}(e_1, e_1^{-1}, e_2, e_2^{-1}) > \mathrm{cxty}(e'_1, {e'}_1^{-1}, e'_2, {e'}_2^{-1})$.

(b) If one of (e_1, e_1^{-1}), (e_2, e_2^{-1}) is not exceptional and they have no ancestor-descendant relationship, then $m(g_1, g_2) \leq \min\{\frac{l(g_1)}{2}, \frac{l(g_2)}{2}\}$ for any $g_1, g_2 \in \{e_1, e_1^{-1}, e_2, e_2^{-1}\}$ unless $g_1 g_2 = I$.

Proof. (a) We first assume that (e_1, e_1^{-1}) is exceptional. Then (e_2, e_2^{-1}) cannot be exceptional and we suppose that $(e_2, e_2^{-1}) = (\epsilon_2 P_2 ABAQ_2, -\epsilon_2 P_2 A^2BA^2Q_2)$.

When $(e_1, e_1^{-1}) = (-A^2B, -BA)$ the pair (e_2, e_2^{-1}) is a vertex of either T_B or T_{BA}. If it is a vertex of T_B, then both words P_2ABAQ_2 and $P_2A^2BA^2Q_2$ end with AB and start with BA. Therefore the following sequence transforms $(e_1, e_1^{-1}, e_2, e_2^{-1})$ into a desired quadruple with a smaller complexity.

$$(-A^2B, -BA, e_2, e_2^{-1}) \longrightarrow (-A^2B, e_2, e_2^{-1}, -BA) \longrightarrow (-A^2B, -BA, A^2Be_2BA, A^2Be_2^{-1}BA).$$

Otherwise, (e_2, e_2^{-1}) is a vertex of T_{BA} and then both P_2ABAQ_2 and $P_2A^2BA^2Q_2$ end with ABA and start with A^2BA. Therefore, the following substitution is desired.

$$(-A^2B, -BA, e_2, e_2^{-1}) \longrightarrow (e_2, e_2^{-1}, -A^2B, -BA) \longrightarrow (-A^2B, -BA, BAe_2A^2B, BAe_2^{-1}A^2B).$$

When $(e_1, e_1^{-1}) = (-ABA, A^2BA^2)$ or $(e_1, e_1^{-1}) = (-BA^2, -AB)$, we have similar arguments.

Now we assume that both (e_1, e_1^{-1}) and (e_2, e_2^{-1}) are unexceptional. Suppose that $(e_i, e_i^{-1}) = (\epsilon_i P_i ABAQ_i, -\epsilon_i P_i A^2BA^2Q_i)$, for $i = 1, 2$. Then Q_2 is extended from Q_1 by a product of finitely many but at least one BA or BA^2 on the left, say

$$Q_2 = (\prod_{i=1}^{\mu} BA^{r_i})Q_1$$

with $\mu \geq 1$ and $r_i \in \{1, 2\}$, for each $i = 1, \ldots, \mu$. Therefore, the following substitution is desired for the case $r_\mu = 1$.

$$\begin{aligned}
&\big(e_1, e_1^{-1}, e_2, e_2^{-1}\big)\\
=\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1, \epsilon_2P_2ABAQ_2, -\epsilon_2P_2A^2BA^2Q_2\big)\\
=\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1,\\
&\quad \epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu} BA^{r_i})Q_1, -\epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu} BA^{r_i})Q_1\big)\\
\to\ &\big(\epsilon_1P_1ABAQ_1,\\
&\quad \epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu} BA^{r_i})Q_1, -\epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu} BA^{r_i})Q_1,\\
&\quad -\epsilon_1P_1A^2BA^2Q_1\big)\\
\to\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1,\\
&\quad \epsilon_2P_1A(\prod_{i=\mu-1}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu-1} BA^{r_i})A^2Q_1, -\epsilon_2P_1A(\prod_{i=\mu-1}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu-1} BA^{r_i})A^2Q_1\big).
\end{aligned}$$

Besides, the following substitution is desired for the case $r_\mu = 2$.

$$\begin{aligned}
&\big(e_1, e_1^{-1}, e_2, e_2^{-1}\big)\\
=\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1, \epsilon_2P_2ABAQ_2, -\epsilon_2P_2A^2BA^2Q_2\big)\\
=\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1,\\
&\quad \epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu} BA^{r_i})Q_1, -\epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu} BA^{r_i})Q_1\big)\\
\to\ &\big(\epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu} BA^{r_i})Q_1, -\epsilon_2P_1(\prod_{i=\mu}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu} BA^{r_i})Q_1,\\
&\quad \epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1\big)\\
\to\ &\big(\epsilon_1P_1ABAQ_1, -\epsilon_1P_1A^2BA^2Q_1,\\
&\quad \epsilon_2P_1A^2(\prod_{i=\mu-1}^{1} A^{3-r_i}B)ABA(\prod_{i=1}^{\mu-1} BA^{r_i})AQ_1, -\epsilon_2P_1A^2(\prod_{i=\mu-1}^{1} A^{3-r_i}B)A^2BA^2(\prod_{i=1}^{\mu-1} BA^{r_i})AQ_1\big).
\end{aligned}$$

(b) When one of (e_1, e_1^{-1}) and (e_2, e_2^{-1}) is exceptional, the other belongs to the unique sub-tree either T_{BA^2}, T_B or T_{BA}. Therefore, $m(g_1, g_2) \leq 1$ unless $g_1g_2 = I$. When (e_1, e_1^{-1}) and (e_2, e_2^{-1}) belong to different sub-trees of T_{BA^2}, T_B and T_{BA}, again we have $m(g_1, g_2) \leq 1$ unless $g_1g_2 = I$.

Now we assume that (e_1, e_1^{-1}) and (e_2, e_2^{-1}) are vertices of the same sub-tree either T_{BA^2}, T_B or T_{BA}. Let $(\epsilon PABAQ, -\epsilon PA^2BA^2Q)$ be the lowest common ancestor of (e_1, e_1^{-1}) and (e_2, e_2^{-1}). Therefore, there exist the reduced forms of e_1, e_1^{-1}, e_2 and e_2^{-1} such that

$$(e_i, e_i^{-1}) = (\epsilon_i P(A^{3-r_i}B)\omega_i(BA^{r_i})Q, -\epsilon_i P(A^{3-r_i}B)\omega_i^{-1}(BA^{r_i})Q)$$

with $r_i \in \{1,2\}$, $\epsilon_i = \pm I$, $\omega_i \in \mathrm{SL}(2,\mathbb{Z})$ and $r_1 \neq r_2$, for $i = 1, 2$. Hence, $m(g_1, g_2) \leq l(Q) + 1$ unless $g_1 g_1 = I$. □

Suppose that a tuple of the form $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ in $\mathrm{SL}(2,\mathbb{Z})$ is a global monodromy of torus achiral Lefschetz fibrations. One can write it as a formal sum $\sum_{v \in V(\Gamma)} m_v \cdot v$ such that $\sum m_v = p$. By Lemma 2.1.3, different tuples which can be written as the same formal sum are Hurwitz equivalent.

Lemma 2.4.7. *Let $\sum_{v \in V(\Gamma)} m_v \cdot v$ be a formal sum over vertices of Γ. Let $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ be a $(2p)$-tuple in $\mathrm{SL}(2,\mathbb{Z})$ expressed by $\sum_{v \in V(\Gamma)} m_v \cdot v$. Suppose that there exist distinct vertices v_1, v_1 such that $m_{v_1} \geq 1$, $m_{v_2} \geq 1$ and there exists an ancestor-descendant relationship between v_1 and v_2. Then, by a sequence of elementary transformations the $(2p)$-tuple can be transformed into a tuple of the form $(e'_1, e'^{-1}_1, \ldots, e'_p, e'^{-1}_p)$ with a smaller complexity.*

Proof. The lemma follows from Lemma 2.4.6 (a). □

On the other hand, we have the following lemma.

Lemma 2.4.8. *Let $\sum_{v \in V(\Gamma)} m_v \cdot v$ be a formal sum over vertices of Γ. Let $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ be a $(2p)$-tuple in $\mathrm{SL}(2,\mathbb{Z})$ expressed by $\sum_{v \in V(\Gamma)} m_v \cdot v$. Suppose that any two distinct vertices v_1, v_2 with $v_1 \geq 1$, $v_2 \geq 1$ have no ancestor-descendant relationship. Then, either*

(i) there exist at least two distinct exceptional vertices with $m_v \geq 1$, or

(ii) there exists at most one of the three exceptional vertices satisfying $m_v \geq 1$.

In Case (i), all components of the $(2p)$-tuple are short (i.e. $\mathrm{cxty}(e_i) = 0$ for $i = 1, \ldots, p$) and by a sequence of elementary transformations the $(2p)$-tuple can be transformed into

$$(-A^2B, -BA) \bullet (-ABA, A^2BA^2)^{p-1}.$$

In Case (ii), the tuple $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ is minimal according to the complexity among tuples obtained from $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ using a sequence of elementary transformations. Besides, all minimal tuples according to the complexity among them of the form $(e'_1, e'^{-1}_1, \ldots, e'_p, e'^{-1}_p)$ are expressed by the formal sum $\sum_{v \in V(\Gamma)} m_v \cdot v$.

Proof. As in Case (i), when there exist at least two distinct exceptional vertices occurring in the product form, by an elementary transformation, the corresponding quadruple can be transformed into

$$(-A^2B, -BA, -ABA, A^2BA^2).$$

Besides, the resulting quadruple contains a generating set of $\mathrm{SL}(2,\mathbb{Z})$. Therefore, by Lemma 2.1.6, the $(2p)$-tuple can be transformed into $(-A^2B, -BA) \bullet (-ABA, A^2BA^2)^{p-1}$, as desired.

In Case (ii), we assume that there exists a sequence of elementary transformations that transforms $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ into a new tuple with a smaller complexity or a new tuple of the form $(e'_1, e'^{-1}_1, \ldots, e'_p, e'^{-1}_p)$ with the same complexity but expressed by a different formal sum of vertices in $V(\Gamma)$. Therefore, there exists at least one component of the new tuple, say $Q^{-1}\omega Q$, where ω is equal to some component of $(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$ and Q is a product of $e_1, \ldots, e_p$ and their inverses, such that $\mathrm{cxty}(Q^{-1}\omega Q) < \mathrm{cxty}(\omega)$. It contradicts Lemma 2.4.6 (b). □

Proof of Theorem 2.4.5. Since each component $g \in \mathrm{SL}(2,\mathbb{Z})$ in a global monodromy of torus achiral Lefschetz fibrations is uniquely determined by $\iota(g) \in \mathrm{PSL}(2,\mathbb{Z})$, by Theorem 2.3.15, torus achiral Lefschetz fibrations of type $\mathcal{O} = [\underbrace{I_1^+, \ldots, I_1^+}_{p \text{ components}}, \underbrace{I_1^-, \ldots, I_1^-}_{q \text{ components}}]$ have pairwise Hurwitz equivalent global monodromies when $p \neq q$.

When $p = q$, each global monodromy is Hurwitz equivalent to a tuple in $\mathrm{SL}(2,\mathbb{Z})$ of the form

$$(e_1, e_1^{-1}, \ldots, e_p, e_p^{-1})$$

and hence can be written as a formal sum $\sum_{v \in V(\Gamma)} m_v \cdot v$. We enumerate all possible formal sums of vertices that express minimal tuples according to the complexity among all Hurwitz equivalent tuples. By Lemma 2.4.7, there is no ancestor-descendant relationship between any two distinct vertices in such a formal sum of vertices.

If there are at least two distinct exceptional vertices v_1 and v_2 such that $m_{v_1} \geq 1$ and $m_{v_2} \geq 1$, by Case (i) in Lemma 2.4.8, then all possible formal sums like this are associated with the same tuple up to Hurwitz equivalence.

If there exists the unique exceptional vertex v occurring in the formal sum, then other vertices belong to the same sub-tree T that is either T_{BA^2}, T_B or T_{BA}. Therefore, by Case (ii) in Lemma 2.4.8, all possible formal sums are in one-to-one correspondence with elements in $\Omega(T, p - m_v)$.

Otherwise, there is no exceptional vertex in the formal sum. Again by Case (ii) in Lemma 2.4.8, all possible formal sums are in one-to-one correspondence with elements in $\Omega(T_{BA^2} \sqcup T_B \sqcup T_{BA}, p)$. $\square$

2.5 Computability of Theorem A

We have included this section to demonstrate that all Hurwitz equivalences occurring in our results (including Theorem A, Theorem B, Theorem C) are computable. For Theorem A, an algorithm exists to provide a sequence of elementary transformations that transforms one tuple to the other. Its time complexity is

$$O\Big(n^5 + n^3 \sum_{i\in\{1,2\}} \sum_{j=1}^{n} l(g_j^{(i)}) + n\big(\sum_{i\in\{1,2\}} \sum_{j=1}^{n} l(g_j^{(i)})\big)^2\Big).$$

We implement the algorithm in C++ and make our code available on GitHub : `https://github.com/AHdoc/monodromy_normalisation`.

The main goal is to analyse the computability of Theorem 2.1.16. In particular, in Theorem 2.3.14, we need an algorithm to make a tuple inverse-free by elementary transformations, but we cannot use Theorem 2.1.16 directly.

Step 1. Suppose that $(g_1, \ldots, g_n)$ is a tuple in $\mathrm{PSL}(2, \mathbb{Z})$ whose components are conjugate to short elements. Recall that Theorem 2.3.14 shortens the tuple by removing some pairs of the form (x, x^{-1}) and triples of the form (l, l, l) with $l^3 = 1$; the resulting tuple is an inverse-free tuple of short elements. The proof uses an induction on an inverse-free tuple, within which one operation seeks to make the sum of $\mathcal{S}$-complexities strictly-smaller by a sequence of elementary transformations. In fact, it is sufficient to check all transformations of the form $(R_i)^t$ with $t \in \{-2, -1, +1, +2\}$.

However, if we throw out the inverse-freeness, the induction still works well and ends with $\mu = 0$, but the restoration operations (see Subsection 2.1.2) cannot result in a tuple of short elements. Indeed, a pair of long elements of the form (x, x^{-1}) could have been combined into a single 1, which was a contradiction in Proposition 2.3.10. Therefore, restorations result in a tuple $(h_1, \ldots, h_m)$ that probably contains sub-pairs $(h_i, h_{i+1}) = (x, x^{-1})$ or/and sub-triples $(h_i, h_{i+1}, h_{i+2}) = (l, l, l)$ with $l^3 = 1$. Using cyclic permutations, we move these pairs and triples, if exist, to the rightmost positions. Hence, we get a tuple of short elements, still denoted by $(h_1, \ldots, h_m)$.

Proposition 2.3.10 asks us to handle each restoration $(h_1 h_2) \dashrightarrow (h_1, h_2)$ carefully. We repeat the search for $(R_i)^t$ with $t \in \{-2, -1, +1, +2\}$ that makes $f(h_i) + f(h_{i+1})$ strictly-smaller. In conclusion, Step 1 calls the following procedure.

---

procedure SHORTEN($(g_1, \ldots, g_n)$)
    $(h_1, \ldots, h_m) \leftarrow (g_1, \ldots, g_n)$, $(k_1, \ldots, k_l) \leftarrow$ empty tuple
    while True **do**   ▷ see Subsection 2.3.2
        if $\exists i$ s.t. Operation 1 is available on $(h_1, \ldots, h_m)$ for i **then**
            combine (h_i, h_{i+1}) into $h_i h_{i+1}$
        else if $\exists i$ s.t. $h_i = 1$ **then**
            $(h_1, \ldots, h_m) \leftarrow (1, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_m)$ ▷ via a cyclic permutation
        else if $\exists i$ and $t \in \{-2, -1, +1, +2\}$ s.t. $(R_i)^t$ makes $\sum_j f(h_j)$ strictly-smaller **then**
            $(h_1, \ldots, h_m) \leftarrow (R_i)^t (h_1, \ldots, h_m)$
        else if $\exists i$ s.t. R_i keeps $\sum_j f(h_j)$ unchanged but makes $l(h_i)$ smaller **then**
            $(h_1, \ldots, h_m) \leftarrow R_i (h_1, \ldots, h_m)$
        else
            break while
        end if
    end while
    while $\exists$ a restoration on $h_i = \widetilde{h_1}\widetilde{h_2}$ **do**

        $(h_1, \dots, h_m) \leftarrow (h_1, \dots, h_{i-1}, \widetilde{h_1}, \widetilde{h_2}, h_{i+1}, \dots, h_m)$
        while True **do**
            if $\exists t \in \{-2, -1, +1, +2\}$ s.t. $(R_i)^t$ makes $\sum_j f(h_j)$ strictly-smaller **then**
                $(h_1, \dots, h_m) \leftarrow (R_i)^t (h_1, \dots, h_m)$
            else
                break while
            end if
        end while
    end while
    while $\exists i$ s.t. $h_i h_{i+1} = 1$ **do**
        $(k_1, \dots, k_l) \leftarrow (h_i, h_{i+1}) \bullet (k_1, \dots, k_l), (h_1, \dots, h_m) \leftarrow (h_1, \dots, h_{i-1}, h_{i+2} \dots, h_m)$
    end while
    while $\exists i$ s.t. $h_i = h_{i+1} = h_{i+2}$ and $h_i h_{i+1} h_{i+2} = 1$ **do**
        $(k_1, \dots, k_l) \leftarrow (h_i, h_{i+1}, h_{i+2}) \bullet (k_1, \dots, k_l), (h_1, \dots, h_m) \leftarrow (h_1, \dots, h_{i-1}, h_{i+3}, \dots, h_m)$
    end while
    return $(h_1, \dots, h_m)$ and $(k_1, \dots, h_l)$
end procedure

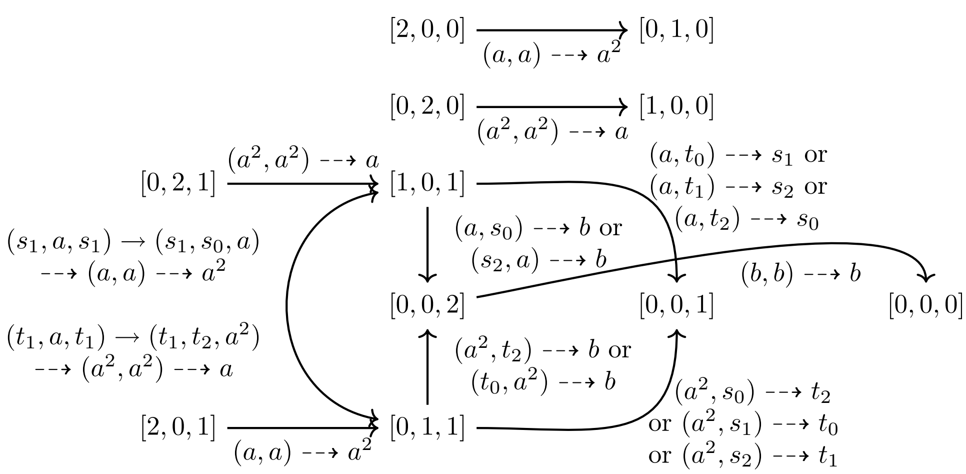
The input of SHORTEN is an arbitrary tuple $(g_1, \dots, g_n)$ in $\mathrm{PSL}(2, \mathbb{Z})$ of conjugates of short elements. The output is the concatenation of a tuple $(h_1, \dots, h_m)$ of short elements and some pairs of the form (x, x^{-1}) and some triples of the form (l, l, l), $l^3 = 1$, say $(h_1, \dots, h_m) \bullet (k_1, \dots, k_l)$. In general, the tuple $(h_1, \dots, h_m)$ is not inverse-free. Therefore the difficulty is inherited to the next step.

Time complexity : A step of the induction in SHORTEN either decreases $\sum_i f(h_i)$ or decreases the number of the pairs (i, j) such that $1 \le i < j \le m$ but $l(h_i) > l(h_j)$. Therefore, the time complexity of SHORTEN($(g_1, \dots, g_n)$) is $O((n^2 + \sum_i l(g_i)) n \sum_i l(g_i))$.

Step 2. The tuple $(h_1, \dots, h_m)$ of short elements probably has two components (resp. three components) that form a tuple of mutually inverse elements (resp. a triple of the form (l, l, l) with $l^3 = 1$). In this case, we move these components to the rightmost positions using cyclic permutations so that $(h_1, \dots, h_m)$ is transformed into the concatenation of a shorter tuple, still denoted by $(h_1, \dots, h_m)$, and a pair (resp. a triple). However, cyclic permutations do not keep components of $(h_1, \dots, h_m)$ short. We end up with this reduction in an extra call on SHORTEN($(h_1, \dots, h_m)$) and then repeat it.

Time complexity : Using two/three cyclic permutations, we transform a tuple of short elements into a tuple, denoted by $(h_1, \dots, h_m)$, such that $\sum_i l(h_i) = O(m)$. Therefore, the above reduction is $O(m(m^2 + m)m^2) = O(m^5)$.

From now on, we can assume that $(h_1, \dots, h_m)$ contains at most 2 components equal to a, at most 2 components equal to a^2, at most 1 component equal to b and a, a^2 cannot appear together within this tuple. We mark a tuple of short elements with c_a components equal to a, c_{a^2} components equal to a^2 and c_b components equal to b with the signature $[c_a, c_{a^2}, c_b]$. The following diagram shows a method to simplify such a tuple into a tuple of signature $[c_a, c_{a^2}, c_b]$ with $c_a + c_{a^2} + c_b \le 1$ (c.f. Step 2, Step 3 and Step 4 in the proof of Theorem 2.1.16).



In the diagram, a reduction from a tuple of signature $[c_a, c_{a^2}, c_b]$ to a tuple of signature $[c'_a, c'_{a^2}, c'_b]$ is a directed edge endowed with some elementary transformations and contractions on a pair or a triple. The reduction starts with cyclic permutations that create a sub-pair or a sub-triple with which the edge is first endowed. It ends with a call on SHORTEN.

Signature $[0,0,2]$ is the only exception that does not satisfy the hypotheses on the tuple. However, with at most 4 contractions, any tuple of short elements satisfying hypotheses can be transformed into a tuple of signature $[c_a, c_{a^2}, c_b]$ with $c_a + c_{a^2} + c_b \leq 1$. Indeed, if a tuple of signature $[1,0,1]$ has to be aimed at a tuple of signature $[0,1,1]$, then it is a tuple of a, b, s_1 and it is transformed into a tuple of a^2, b, s_0, s_1, s_2 of signature $[0,1,1]$, which is further transformed into a tuple of signature $[0,0,1]$.

Time complexity : Both cyclic permutation and contraction are linear. We have shown that a cyclic permutation on a tuple of short elements results in a tuple such that $\sum_i l(h_i) = O(m)$. The simplification along the diagram calls SHORTEN at most 4 times, therefore its time complexity is $O((m^2+m)m^2) = O(m^4)$.

From now on, we can further assume that $(h_1, \ldots, h_m)$ contains at most 1 component equal to either a, a^2 or b. Proposition 2.3.4 claims that, if $(h_1, \ldots, h_m)$ is inverse-free, then it is either a tuple of a, a^2, b, s_0, s_1, s_2 or a tuple of a, a^2, b, t_0, t_1, t_2. The proof is a rearrangement of s_0, s_1, s_2 and t_0, t_1, t_2. To introduce a similar reduction, we provide the following procedure.

---

procedure ST-REARRANGEMENT$((\kappa_1, \ldots, \kappa_l))$
  if $\exists i$ s.t. $\kappa_i \in \{a, a^2, b\}$ **then**
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_i, \kappa_{i+1}, \ldots, \kappa_l, \kappa_1, \ldots, \kappa_{i-1})$ ▷ via a cyclic permutation
  end if
  while $\exists i$ s.t. $\kappa_i \in \{t_0, t_1, t_2\}$ and $\kappa_{i+1} \in \{s_0, s_1, s_2\}$ **do**
    if $(\kappa_i, \kappa_{i+1}) \in \{(t_0, s_0), (t_1, s_1), (t_2, s_2)\}$ **then**
      return $(\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+2}, \ldots, \kappa_l)$
    else if $(\kappa_i, \kappa_{i+1}) \in \{(t_0, s_1), (t_1, s_2), (t_2, s_0)\}$ **then**
      $(\kappa_1, \ldots, \kappa_l) \leftarrow R_i^{-1}(\kappa_1, \ldots, \kappa_l)$
    else if $(\kappa_i, \kappa_{i+1}) \in \{(t_0, s_2), (t_1, s_0), (t_2, s_1)\}$ **then**
      $(\kappa_1, \ldots, \kappa_l) \leftarrow R_i(\kappa_1, \ldots, \kappa_l)$
    end if
  end while
  if $\exists i < j$ s.t. $(\kappa_i, \kappa_j) \in \{(s_0, t_0), (s_1, t_1), (s_2, t_2)\}$ **then**
    return $(\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+1}^{\kappa_j}, \ldots, \kappa_{j-1}^{\kappa_j}, \kappa_{j+1}, \ldots, \kappa_l)$
  else if $\exists i < j < k$ s.t. $(\kappa_i, \kappa_j, \kappa_k) \in \{(s_0, t_1, t_2), (s_1, t_2, t_0), (s_2, t_0, t_1)\}$ **then**
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (R_2)^2(\kappa_i, \kappa_j, \kappa_k, \kappa_1^{\kappa_i\kappa_j\kappa_k}, \ldots, \kappa_{i+1}^{\kappa_j\kappa_k}, \ldots, \kappa_{j+1}^{\kappa_k}, \ldots, \kappa_{k+1}, \ldots)$
    return $(\kappa_3, \ldots, \kappa_l)$
  else if $\exists i < j < k$ s.t. $(\kappa_i, \kappa_j, \kappa_k) \in \{(s_2, s_1, t_0), (s_0, s_2, t_1), (s_1, s_0, t_2)\}$ **then**
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (R_2)^{-1}(\kappa_i, \kappa_j, \kappa_k, \kappa_1^{\kappa_i\kappa_j\kappa_k}, \ldots, \kappa_{i+1}^{\kappa_j\kappa_k}, \ldots, \kappa_{j+1}^{\kappa_k}, \ldots, \kappa_{k+1}, \ldots)$
    return $(\kappa_3, \ldots, \kappa_m)$
  else
    return $(\kappa_1, \ldots, \kappa_m)$
  end if
end procedure

The input of ST-REARRANGEMENT is a tuple of short elements, say $(\kappa_1, \ldots, \kappa_l)$, that contains at most 1 component equal to a, a^2 or b. The output is either a tuple of short elements of length l, meaning that $(\kappa_1, \ldots, \kappa_l)$ is inverse-free, or a tuple $(\widetilde{\kappa_1}, \ldots, \widetilde{\kappa_{l-2}})$ of length $l-2$, meaning that $(\kappa_1, \ldots, \kappa_l)$ can be transformed into $(\widetilde{\kappa_1}, \ldots, \widetilde{\kappa_{l-2}}) \bullet (s_i, t_i)$ with some i by elementary transformations.

We call ST-REARRANGEMENT and SHORTEN with $(h_1, \ldots, h_m)$ repeatedly unless the tuple is inverse-free. In conclusion, Step 2 calls a procedure, named as INVERSE-FREE, whose input is a tuple $(h_1, \ldots, h_m)$ of short elements and output is an inverse-free tuple $(\kappa_1, \ldots, \kappa_l)$ of short elements such that at most 1 component is equal to a, a^2 or b.

Time complexity : The procedure ST-REARRANGEMENT decreases the length of the tuple and transform a tuple of short elements into a tuple, denoted by $(\kappa_1, \ldots, \kappa_l)$, such that $\sum_i l(\kappa_i) = O(l)$. The time complexity of ST-REARRANGEMENT is $O(l^4)$.

Meanwhile, SHORTEN transforms the tuple back to a tuple of short elements. In conclusion, the time complexity of INVERSE-FREE$(h_1, \ldots, h_m)$ is $O(m^5+m^4+m(m^4+((m^2+m)m^2))) = O(m^5)$.

Step 3. To slightly improve the complement to Theorem of R. Livné introduced in [Moi77, p.180-187] to a tuple of a, b, s_0, s_1, s_2 that at most 1 component is equal to a or b, we first introduce the following procedure named as MOISHEZON (c.f. Proposition 2.3.6).

---

procedure MOISHEZON$((\kappa_1, \ldots, \kappa_l))$

if $\exists i$ s.t. $\kappa_i \in \{a, b\}$ **then**
  $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_i, \kappa_{i+1}, \ldots, \kappa_l, \kappa_1, \ldots, \kappa_{i-1})$ ▷ via a cyclic permutation
end if
while True **do**
  if $\exists i$ s.t. $(\kappa_i, \kappa_{i+1}) = (s_1, s_0)$ **then** ▷ decrease # of s_1
    $(\kappa_1, \ldots, \kappa_l) \leftarrow R_i(\kappa_1, \ldots, \kappa_l)$
  else if $\exists i$ s.t. $(\kappa_i, \kappa_{i+1}) = (s_2, s_1)$ **then** ▷ decrease # of s_1
    $(\kappa_1, \ldots, \kappa_l) \leftarrow R_i^{-1}(\kappa_1, \ldots, \kappa_l)$
  else if $\exists i$ s.t. $(\kappa_i, \kappa_{i+1}, \kappa_{i+2}) = (s_0, s_2, s_0)$ **then** ▷ Claim 1
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-1}, s_2, s_0, s_2, \kappa_{i+3}, \ldots, \kappa_l)$
  else if $\exists i + 1 < j$ s.t. $(\kappa_i, \ldots, \kappa_{j+1}) = (s_1, s_2, \ldots, s_2, s_0, s_2)$ **then** ▷ Claim 2
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-1}, s_1, s_0, s_2, s_0 \ldots, s_0, \kappa_{j+2}, \ldots, \kappa_l)$
  else if $\exists i + 2 < j$ s.t $(\kappa_{i-1}, \ldots, \kappa_{j+1}) = (s_2, s_0, s_2, \ldots, s_2, s_0, s_2)$ **then** ▷ Claim 3
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-2}, s_2, s_0, s_2, s_0, s_2, s_0, \ldots, s_0, \kappa_{j+2}, \ldots, \kappa_l)$
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-2}, s_0, \ldots, s_0, \kappa_{j+2}, \ldots, \kappa_l)$
  else if $\exists i$ s.t. $(\kappa_i, \ldots, \kappa_{i+5}) = (s_0, s_2, s_2, s_0, s_2, s_2)$ **then** ▷ Claim 4
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+6}, \ldots, \kappa_l)$
  else if $\exists i$ s.t. $(\kappa_i, \ldots, \kappa_{i+5}) = (s_2, s_2, s_0, s_2, s_2, s_0)$ **then** ▷ Claim 4
    $(\kappa_1, \ldots, \kappa_l) \leftarrow (\kappa_1, \ldots, \kappa_{i-1}, \kappa_{i+6}, \ldots, \kappa_l)$
  else
    break while
  end if
end while
end procedure

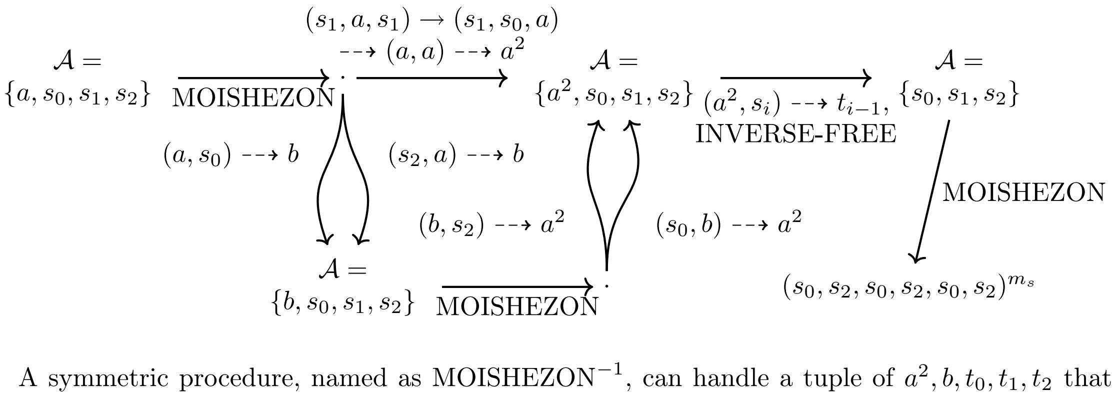
The input of MOISHEZON is an inverse-free tuple $(\kappa_1, \ldots, \kappa_l)$ of a, b, s_0, s_1, s_2 that at most 1 component is equal to a or b. The output, denoted by $(\kappa'_1, \ldots, \kappa'_{l'})$, is again a tuple of a, b, s_0, s_1, s_2 and shows that $(\kappa_1, \ldots, \kappa_l)$ can be transformed into the concatenation of $(\kappa'_1, \ldots, \kappa'_{l'})$ and some sextuples of the form $(s_0, s_2, s_0, s_2, s_0, s_2)$ by elementary transformations. If $(\kappa_1, \ldots, \kappa_l)$ is a tuple of s_0, s_1, s_2, then $l' = 0$; otherwise, by Lemma 2.3.18 and Lemma 2.3.20, either

- $(\kappa'_1, \kappa'_2) = (a, s_0)$ or $(\kappa'_l, \kappa'_1) = (s_2, a)$ or $(\kappa'_l, \kappa'_1, \kappa'_2) = (s_1, a, s_1)$, or
- $(\kappa'_1, \ldots, \kappa'_l)$ starts with (b, s_2) or $(b) \bullet (s_0)^{v_{0,1}} \bullet (s_2, s_0)$ with $v_{0,1} \geq 1$, or
- $(\kappa'_2, \ldots, \kappa'_l, \kappa'_1)$ ends with (s_0, b) or $(s_2, s_0) \bullet (s_2)^{u_{\mu, n_\mu}} \bullet (b)$ with $u_{\mu, n_\mu} \geq 1$.

By elementary transformations and at most 2 contractions, the tuple $(\kappa'_1, \ldots, \kappa'_{l'})$ is further transformed into a tuple of a, a^2, b, s_0, s_1, s_2 that at most 1 component is equal to a, a^2 or b.

Time complexity : The procedure MOISHEZON($(\kappa_1, \ldots, \kappa_l)$) is looping, seeks the minimal number of components equal to s_1 and seeks the minimal according to the lexicographical order given by $s_0 < s_2$. Therefore, the number of times that the loop loops is related to the number of reverse pairs, i.e. $i < j$ but $\kappa_i > \kappa_j$ according to the lexicographical order, which is $O(l^2)$. The time complexity of MOISHEZON is $O(l^5)$.

In Step 3, we consider an inverse-free tuple $(\kappa_1, \ldots, \kappa_l)$ of short elements that contains at most 1 component equal to a, a^2 or b. Let $\mathcal{A}$ be the set of elements in $(\kappa_1, \ldots, \kappa_l)$. We follow the diagram below to reduce the tuple using elementary transformations and at most 3 contractions.



A symmetric procedure, named as MOISHEZON^{-1}, can handle a tuple of a^2, b, t_0, t_1, t_2 that at most 1 component is equal to a^2 or b. Therefore, we have a symmetric diagram for the rest of the cases.

Time complexity : In conclusion, Step 3 contracts the tuple at most 3 times, calls MOISHEZON several times and calls INVERSE-FREE at most once. Its time complexity is $O(l^5)$.

Step 4. We have shown that by elementary transformations and at most 7 contractions, the initial tuple $(g_1,\ldots,g_n)$ is transformed into a concatenation of the following tuples.

$$(s_1,t_1)^Q, (t_1,s_1)^Q, (a,a^2)^Q, (a^2,a)^Q, (b,b)^Q, (a,a,a)^Q, (a^2,a^2,a^2)^Q,$$
$$(s_0,s_2,s_0,s_2,s_0,s_2), (t_0,t_2,t_0,t_2,t_0,t_2)$$

where $Q \in \mathrm{PSL}(2,\mathbb{Z})$ is arbitrary, such that $(s_0,s_2,s_0,s_2,s_0,s_2)$ and $(t_0,t_2,t_0,t_2,t_0,t_2)$ cannot appear at the same time. By Lemma 2.1.3, the concatenation can be transformed into such with a desired order by elementary transformations. Besides, a pair of the form (x,x^{-1}) can be transformed into (x^{-1},x) by an elementary transformation. Therefore, we handle each restoration carefully and obtain

$$(h_1,\ldots,h_m)\bullet(s_0,s_2)^{3m_s}\bullet(t_0,t_2)^{3m_t}\bullet\prod_{i=1}^{m_{st}}(s_1,t_1)^{X_i}\bullet\prod_{i=1}^{m_a}(a,a^2)^{Y_i}\bullet\prod_{i=1}^{m_b}(b,b)^{Z_i}\bullet\prod_{\epsilon=\pm1}\prod_{i=1}^{n_\epsilon}(a^\epsilon,a^\epsilon,a^\epsilon)^{P_{\epsilon,i}}$$

with $m_s m_t = 0$, $X_i, Y_i, Z_i, P_{\epsilon,i} \in \mathrm{PSL}(2,\mathbb{Z})$, which is Hurwitz equivalent to the initial tuple.

The tuple $(h_1,\ldots,h_m)$, called the exceptional part of the resulting tuple, is a tuple of short elements. In fact, if $p=q$, $|p_a-q_a|\equiv 0 \pmod 3$ and n_b is even in Theorem 2.1.16, the exceptional part does not exist anymore, i.e. $m=0$, thus we have already finished the computation. Otherwise, the exceptional tuple $(h_1,\ldots,h_m)$ contains a generating set ; by Lemma 2.1.6, we obtain

$$(h_1,\ldots,h_m)\bullet(s_0,s_2)^{3m_s}\bullet(t_0,t_2)^{3m_t}\bullet(s_1,t_1)^{m_{st}}\bullet(a,a^2)^{m_a}\bullet(b,b)^{m_b}\bullet(a,a,a)^{n_1}\bullet(a^2,a^2,a^2)^{n_{-1}}.$$

With a slight adjustment using cyclic permutations, we may further assume that $n_1 n_{-1} = 0$.

The length of the exceptional tuple is bounded by a constant. In fact, we claim that the exceptional tuple $(h_1,\ldots,h_m)$ satisfies $m \leq 13$ without further discussion. The proof of Theorem 2.1.16 has revealed that a partial normal form can be transformed into the desired normal form by cyclic transformations and elementary transformations that keep each component short. The whole computation ends with a brute-force search.

Time complexity : The brute-force search is $O(1)$ as the length of the exceptional tuple is bounded by a constant. The time complexity of Step 4 is $O(n\sum_i l(g_i)+n^3+1)$. Hence, the computation of Theorem 2.1.16 has the time complexity $O(n^5+n^3\sum_i l(g_i)+n(\sum_i l(g_i))^2)$.

Chapitre 3

Holomorphic fibrations and holomorphic curves in the moduli space

3.1 Teichmüller theory

We fix non-negative integers g, n and h with $2g-2+n>0$ and $h \geq 2$ once and for all.

3.1.1 Teichmüller space

Let $\Sigma_{g,n}$ be an oriented smooth surface of genus g with n punctures without boundary. The *geometric intersection number* of two closed curves γ_1 and γ_2, denoted by $\iota(\gamma_1, \gamma_2)$, is the minimum cardinality of $\nu_1 \cap \nu_2$ for all closed curves ν_1, ν_2 such that γ_i is homotopic to ν_i, for $i = 1, 2$.

We say that a set of closed curves $\{\gamma_1, \dots, \gamma_k\}$ *fills up* the surface $\Sigma_{g,n}$ if, for any non-contractible non-peripherial closed curve ν, $\iota(\nu, \gamma_i) \geq 1$ for some i. A set of disjoint simple closed curves on $\Sigma_{g,n}$ is called a multi-curve. By convention, we define the product $\gamma_1 \cdot \gamma_2$ of two oriented paths as their concatenation and the inverse γ^{-1} of an oriented path is the same path with the opposite orientation.

The Teichmüller space $\mathcal{T}_{g,n}$ consists of all marked Riemann surfaces of type (g, n), i.e. equivalent pairs (X, f_X) where X is a Riemann surface and $f_X : \Sigma_{g,n} \to X$ is an orientation preserving diffeomorphism. Two pairs (X, f_X), (Y, f_Y) are equivalent if $f_Y \circ f_X^{-1} : X \to Y$ is isotopic to a biholomorphism. Based on the uniformization of Riemann surfaces, each equivalent class is represented by a marked hyperbolic surface. Then a mapping class $[\phi] \in \mathrm{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by $[\phi] \cdot [(X, f_X)] = [(X, f_X \circ \phi^{-1})]$. The mapping class group acts properly discontinuously on the Teichmüller space and the quotient space is called the moduli space, denoted by $\mathcal{M}_{g,n}$.

For any two points $[(X, f_X)]$, $[(Y, f_Y)] \in \mathcal{T}_{g,n}$, we define the *Teichmüller distance* $d_{\mathcal{T}}$ by

$$d_{\mathcal{T}}([(X, f_X)], [(Y, f_Y)]) = \frac{1}{2} \log \inf_{\phi} \{K(\phi)\}. \tag{3.1}$$

Here, the infimum is taken over all quasiconformal diffeomorphisms $\phi : X \to Y$ homotopic to $f_Y \circ f_X^{-1}$, i.e. all quasiconformal diffeomorphisms respecting the markings. Moreover, $K(\phi) \geq 1$ denotes dilatation of ϕ.

The *geodesic length function* $L_\gamma([(X, f_X)])$ assigns to each closed curve $\gamma \subset \Sigma_{g,n}$ the length of the unique geodesic homotopic to $f_X(\gamma)$ on the hyperbolic representative X of $[(X, f_X)] \in \mathcal{T}_{g,n}$. The length of a closed curve γ on a hyperbolic surface X is denoted by $l_X(\gamma)$. The next lemma is due to Wolpert.

Lemma 3.1.1 ([Wol79])**.** *Consider two points in $\mathcal{T}_{g,n}$ that are represented by the marked hyperbolic surfaces (X, f_X) and (Y, f_Y). Set $K = \exp\big(2d_{\mathcal{T}}([(X, f_X)], [(Y, f_Y)])\big)$. Then, the geodesic length function is distorted by a factor of at most K, i.e.*

$$\frac{1}{K} L_\gamma([(X, f_X)]) \leq L_\gamma([(Y, f_Y)]) \leq K L_\gamma([(X, f_X)])$$

for any closed curve $\gamma \subset \Sigma_{g,n}$.

Recall the classification of mapping classes due to Bers ([Ber78]). Let $\varphi \in \mathrm{Mod}_h$ be a mapping class. The *translation distance* of φ is defined by

$$\tau(\varphi) = \inf_{[(X,f_X)] \in \mathcal{T}_h} d_{\mathcal{T}}([(X, f_X)], \varphi \cdot [(X, f_X)]).$$

Then $\tau(\varphi) = 0$ and the infimum is attained if and only if φ is periodic. Also $\tau(\varphi)$ is positive and the infimum is attained if and only if φ is pseudo-Anosov. Eventually, $\tau(\varphi)$ is not attained if and only if φ is reducible and of infinite order. Furthermore, if $\tau(\varphi) = 0$ and $\tau(\varphi)$ is not attained, then then there exists $\mu \in \mathbb{Z}_{\geq 1}$ bounded above by a constant determined by h such that ϕ^μ is a multi-twist.

3.1.2 Monodromy homomorphisms

Suppose that $B = \Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface of type (g, n) with some lattice $\Gamma \leq \mathrm{Aut}(\mathbb{H}^2)$. Consider a holomorphic map $F : B \to \mathcal{M}_h$ and the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$. We introduce the induced *monodromy homomorphisms*.

Firstly, the map $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ induces a group homomorphism $F_\Gamma : \Gamma \to \mathrm{Mod}_h$ such that $\widetilde{F} \circ \phi = F_\Gamma(\phi) \circ \widetilde{F}$, for every $\phi \in \Gamma$. When $F(t)$ has non-identical automorphisms (i.e. $F(t)$ is symmetric) for some $t \in B$, the homomorphism F_Γ is not necessarily unique. There are at most $(2g + n)^{84(h-1)}$ many possibilities of such a homomorphism F_Γ.

Secondly, fixing a base point $t \in B$ and lifting it to some $\widetilde{t} \in \mathbb{H}^2$, we obtain a standard group isomorphism $\rho_{t,\widetilde{t}} : \pi_1(B, t) \to \Gamma$ as follows. A loop $\gamma \subset B$ based at t is lifted to the path in $\mathbb{H}^2$ joining $\widetilde{t}$ and $\rho_{t,\widetilde{t}}([\gamma]) \cdot \widetilde{t}$.

Eventually, $F_* := F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$ is called a *monodromy homomorphism* of F. A different choice of $\widetilde{t}$ may change the monodromy homomorphism by a conjugacy.

The monodromy homomorphism enables us to reformulate and slightly improve the most important property of the holomorphic map F, say being distance-decreasing for the intrinsic Kobayashi distances, as follows.

Proposition 3.1.2. *Let $\gamma \subset B$ be a loop based at t. Then* $(1/2)l_B(\gamma) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), F_*([\gamma]) \cdot \widetilde{F}(\widetilde{t})\Big)$.

Proof. By definition, we get

$$\begin{aligned}\frac{1}{2}l_B(\gamma) &= \frac{1}{2}d_{\mathbb{H}^2}\Big(\widetilde{t}, \rho_{t,\widetilde{t}}([\gamma])(\widetilde{t})\Big) \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), \widetilde{F} \circ \rho_{t,\widetilde{t}}([\gamma_i])(\widetilde{t})\Big) \\ &= d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), \big(F_\Gamma \circ \rho_{t,\widetilde{t}}([\gamma])\big) \circ \widetilde{F}(\widetilde{t})\Big) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t}), F_*([\gamma]) \cdot \widetilde{F}(\widetilde{t})\Big).\end{aligned}$$

□

Proposition 3.1.3. *Let $\gamma \subset B$ be a loop based at t and $\gamma' \subset B$ be a free loop homotopic to γ. Then $(1/2)l_B(\gamma') \geq \tau(F_*([\gamma]))$.*

Proof. Let $H : [0, 1] \times [0, 1] \to B$ be the homotopy between γ and γ' such that $H(0, \cdot) = \gamma(\cdot)$, $H(0, 0) = H(0, 1) = t$ and $H(1, \cdot) = \gamma'(\cdot)$. Based on the path $H(\cdot, 0)$ joining t to $t' := H(1, 0)$, we obtain a lift of t', denoted by $\widetilde{t'}$. Consider the new monodromy homomorphism $F'_* = F_\Gamma \circ \rho_{t',\widetilde{t'}} : \pi_1(B, t') \to \mathrm{Mod}_h$. Since $F'_*([\gamma']) = F_*([\gamma]) =: \phi$, by Proposition 3.1.2, we have

$$\frac{1}{2}l_B(\gamma') \geq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), F'_*([\gamma']) \cdot \widetilde{F}(\widetilde{t'})\Big) \geq \tau(\phi).$$

□

Corollary 3.1.4. *Each peripheral monodromy ϕ satisfies $\tau(\phi) = 0$.*

Proof. The corollary follows from Proposition 3.1.3. □

3.1.3 Essentially purely pseudo-Anosov monodromy

We introduce the following hypothesis on the monodromy homomorphism (see also [Rei06]).

Definition 3.1.5. Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a holomorphic map. We say that a monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$ of F is *essentially purely pseudo-Anosov* if for each non-trivial non-peripheral class $[\gamma] \in \pi_1(B, t)$ the image $F_*([\gamma])$ is pseudo-Anosov.

Isometrically immersed holomorphic curves are specific examples whose monodromy homomorphisms are essentially purely pseudo-Anosov. In fact, as mentioned in [EM11], every closed geodesic in $\mathcal{M}_h$ is the unique loop of minimal length in its homotopy class. We include a proof for the sake of completeness.

Theorem 3.1.6. *Each monodromy homomorphism of a Teichmüller curve is essentially purely pseudo-Anosov.*

Proof. Let $F : B \to \mathcal{M}_h$ be a holomorphic isometric immersion and $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ be a lift of F which is an isometric embedding. Fixing $t \in B$ and lifting it to $\widetilde{t} \in \mathbb{H}^2$, we take a monodromy homomorphism $F_* = F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B,t), \mathrm{Mod}_h)$.

Let $\gamma \subset B$ be a non-trivial non-peripheral loop based at t. Suppose that $\gamma' \subset B$ is a closed geodesic homotopic to γ. Consider the homotopy $H : [0,1] \times [0,1] \to B$ such that $H(0,\cdot) = \gamma(\cdot)$, $H(0,0) = H(0,1) = t$ and $H(1,\cdot) = \gamma'(\cdot)$. Based on the path $H(\cdot,0)$ joining t to some $t' \in \Gamma'$, we obtain a monodromy homomorphism $F'_* = F_\Gamma \circ \rho_{t',\widetilde{t'}} \in \mathrm{Hom}(\pi_1(B,t'), \mathrm{Mod}_h)$ such that $F_*([\gamma]) = F'_*([\gamma']) =: \phi$. By Proposition 3.1.2, we get

$$\frac{N}{2} l_B(\gamma') = \frac{1}{2} d_{\mathbb{H}^2}\Big(\widetilde{t'}, \rho_{t',\widetilde{t'}}([\gamma'])^N(\widetilde{t'})\Big) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big),$$

for any integer $N \geq 1$.

We claim that $\tau(\phi) = d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) = \frac{1}{2} l_B(\gamma')$. Indeed, assume that there exists $\widetilde{q} \in \mathcal{T}_h$ such that $d_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big) < d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big)$. Suppose that N is large enough such that $N\Big(d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) - d_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big)\Big) > 2 d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big)$. Therefore, we get

$$\begin{aligned}
\frac{N}{2} l_B(\gamma') &= d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big) \leq d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big) + d_{\mathcal{T}}\Big(\widetilde{q}, \phi^N \cdot \widetilde{q}\Big) + d_{\mathcal{T}}\Big(\phi^N \cdot \widetilde{q}, \phi^N \cdot \widetilde{F}(\widetilde{t'})\Big) \\
&\leq 2 d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \widetilde{q}\Big) + N d_{\mathcal{T}}\Big(\widetilde{q}, \phi \cdot \widetilde{q}\Big) < N d_{\mathcal{T}}\Big(\widetilde{F}(\widetilde{t'}), \phi \cdot \widetilde{F}(\widetilde{t'})\Big) = \frac{N}{2} l_B(\gamma'),
\end{aligned}$$

a contradiction. Hence, the translation distance of the monodromy along any non-trivial non-peripheral loop is positive. ☐

The following proposition shows that being essentially purely pseudo-Anosov is a sufficiently strong hypothesis on the monodromy homomorphism of a holomorphic map.

Proposition 3.1.7. *Let $B = \Gamma \backslash \mathbb{H}^2$ be an oriented hyperbolic surface of type (g,n). Let $F : B \to \mathcal{M}_h$ be a holomorphic map with an essentially purely pseudo-Anosov monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B,t), Mod_h)$. Then*

(i) F is non-constant;

(ii) the monodromy homomorphism is injective;

(iii) each peripheral monodromy is of infinite order;

(iv) peripheral monodromies of any two cusps are not disjoint along any geodesic segment κ between the boundaries of their cusp regions;

(v) $\mathrm{sys}(B) \geq 2\log 2/(12h-12)$.

Proof. For (i), we notice that there exists at least one non-trivial non-peripheral element in $\pi_1(B,t)$. For (ii), it suffices to show that each peripheral element cannot be represented by the identity. Take the group presentation

$$\pi_1(B,t) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid \prod_i [a_i, b_i] \prod_j c_j = 1 \rangle.$$

with loops $a_1, \ldots, a_g$, $b_1, \ldots, b_g$,$c_1, \ldots, c_n$ at t. Consider a positive power of a peripheral generating loop, say c_j^r with $j = 1, \ldots, n$. When $g \geq 1$ and $n \geq 1$, then $[c_j^r, a_1] \neq 1$ is non-peripheral. When $n \geq 2$, take $j' \neq j$ and then $[c_j^r, c_{j'}] \neq 1$ is non-peripheral. In both cases, we get $F_*(c_j^r) \neq 1$. For (iii), a peripheral monodromy ϕ must be of infinite order due to the injectivity of the monodromy homomorphism.

For (iv), let U_1, U_2 be two distinct cusp regions of B linked by a geodesic segment κ. Set $t_0 = \partial U_1 \cap \kappa$ and take an arbitrary path γ joining t to t_0. The loop along $\gamma \cup \partial U_1$ based at t that goes once around U_1 clockwise is denoted by γ_1 and its monodromy is denoted by ϕ_1. The loop along $\gamma \cup \kappa \cup \partial U_2$ based at t that goes once around U_2 clockwise is denoted by γ_2 and its monodromy is denoted by ϕ_2. By Corollary 3.1.4, some power $\phi_1^{\mu_1}$ is the multi-twist along a multi-curve $\boldsymbol{\alpha_1}$ and some power $\phi_2^{\mu_2}$ is the multi-twist along a multi-curve $\boldsymbol{\alpha_2}$. Assume that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is a set of disjoint simple closed curves. We notice that $\gamma_1^{2\mu_1} \cdot \gamma_2^{2\mu_2}$ is non-peripheral of which the monodromy is reducible, which is a contradiction. The parameters 2 in $\gamma_1^{2\mu_1} \cdot \gamma_2^{2\mu_2}$ cannot be replaced with 1. Indeed, when $(g, n) = (0, 3)$ and $\mu_1 = \mu_2 = 1$, $\gamma_1 \cdot \gamma_2$ is peripheral but $\gamma_1^2 \cdot \gamma_2^2$ not.

For (v), consider an essential closed geodesic $\gamma \subset B$. Then there exists a loop γ' based at t homotopic to γ, which is non-trivial and non-peripheral. By Proposition 3.1.3, we have $(1/2)l_B(\gamma) \geq \tau(F_*([\gamma']))$. Penner proved in [Pen91, p.444] the inequality $\tau(\phi) \geq \log 2/(12h-12)$, for any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}_h$ (see also [FM11, Theorem 14.10]). Thus, $\mathrm{sys}(B) \geq 2\log 2/(12h-12)$. □

We can apply Theorem E - (ii) and Theorem F to a holomorphic map $F : B \to \mathcal{M}_h$ with an essentially purely pseudo-Anosov monodromy homomorphism. Then F is a quasi-isometric immersion with parameters depending only on (g, n) and h. In addition, the lift $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ restricted to any fundamental convex polygon D with exactly n ideal points is a quasi-isometric embedding.

Moreover, by Theorem G, there are only finitely many essentially purely pseudo-Anosov monodromy homomorphisms induced by Teichmüller curves of type (g, n) in $\mathcal{M}_h$, up to equivalence. When $n = 0$, this finiteness is a consequence of Bowditch's result [Bow09] which shows that there are only finitely many conjugacy classes of purely pseudo-Anosov surface subgroups of Mod_h of genus g (see also [Bow17 ; DF09]). In conclusion,

Conjecture 3.1.8. *There are only finitely many conjugacy classes of essentially purely pseudo-Anosov subgroups of* Mod_h *isomorphic to the fundamental group of* $\Sigma_{g,n}$.

We end with the proof of Corollary 1.2.8 as a consequence of the Rigidity Theorem in [IS88].

Proof of Corollary 1.2.8. This comes from Theorem 3.1.6, Proposition 3.1.7 and Theorem G. □

3.1.4 Mumford's compactness

Let $\epsilon > 0$ be an arbitrary real number. For the Teichmüller metric, the moduli space $\mathcal{M}_{g,n}$ has an infinite diameter. In [Mum71], however, Mumford introduced that the ϵ-thick part of the moduli space that consists of hyperbolic surfaces X with $\mathrm{sys}(X) \geq \epsilon$ is compact.

Let $\mathcal{T}_{g,n}^{\geq\epsilon}$ be the set of equivalent classes of marked hyperbolic surfaces whose systole is bounded below by ϵ. The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{T}_{g,n}$ preserves the systole and therefore we take the quotient space of the thick part, denoted by $\mathcal{M}_{g,n}^{\geq\epsilon}$. We have Mumford's compactness.

Theorem 3.1.9 (Mumford)**.** *The ϵ-thick part $\mathcal{M}_{g,n}^{\geq\epsilon}$ of the moduli space $\mathcal{M}_{g,n}$ is a compact subset.*

Form now on, we fix a base point $[(X_0, f_{X_0})]$ in $\mathcal{T}_{g,n}$ that is represented by a marked complete hyperbolic surface (X_0, f_{X_0}). We fix a base point $s \in \Sigma_{g,n}$ and fix the oriented loops

$$\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{g,1}, \gamma_{g,2}, \gamma_1', \ldots, \gamma_n' \subset \Sigma_{g.n}$$

at s as in Figure 3.1.

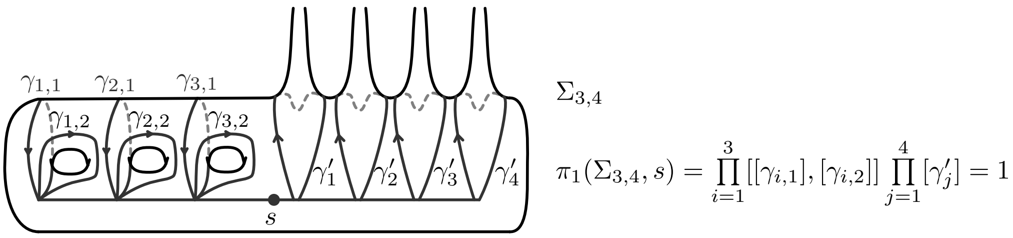


FIGURE 3.1 – The standard loops of $\Sigma_{3,4}$ generate the fundamental group $\pi_1(\Sigma_{g,n}, s)$.

Loops $\gamma_{i,j}$ and γ_k', for $i = 1, \ldots, g$, $j = 1, 2$ and $k = 1, \ldots, n$, are called the *standard loops* of $\Sigma_{g,n}$. They meet the following conditions.

i. $\iota(\gamma_{i,1}, \gamma_{i,2}) = 1$, for each i, but the intersection number of any other two distinct loops is 0 ;
ii. γ_j' goes round the j-th puncture exactly once clockwise, for each j ;

iii. the fundamental group $\pi_1(\Sigma_{g,n}, s)$ is generated by classes of these loops with the relation

$$\prod_{i=1}^{g}[[\gamma_{i,1}], [\gamma_{i,2}]] \prod_{j=1}^{n}[\gamma'_j] = 1.$$

In contrast, a collection of loops $\widehat{\gamma_{i,1}}$, $\widehat{\gamma_{i,2}}$ $\widehat{\gamma'_j} \subset \Sigma_{g,n}$, for $i = 1, \ldots, g$, $j = 1, \ldots, n$, based at some point $\widehat{s} \in \Sigma_{g,n}$ satisfying conditions i, ii, iii implies an orientation preserving diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ such that $\phi(s) = \widehat{s}$ and each $\phi(\gamma_{i,1})$ (resp. $\phi(\gamma_{i,2})$, $\phi(\gamma_j)$) is homotopic to $\widehat{\gamma_{i,1}}$ (resp. $\widehat{\gamma_{i,2}}$, $\widehat{\gamma'_j}$) relative to $\widehat{s}$.

Set $t_0 = f_{X_0}(s) \in X_0$. The set $\{[f_{X_0}(\gamma)] \mid \gamma \text{ is a standard loop}\}$ forms a generating set of $\pi_1(X_0, t_0)$ such that the length of each $f_{X_0}(\gamma)$ is determined by (g, n). In other words, there exists a constant $N(g, n)$ that depends only on (g, n) such that

$$l_{X_0}(f_{X_0}(\gamma)) \leq N(g, n)$$

for each standard loop γ of $\Sigma_{g,n}$. We further notice the following lemma.

Lemma 3.1.10. *Suppose that (Y, f_Y) is a marked hyperbolic surface and $\psi : X_0 \to Y$ is a K-quasiconformal diffeomorphism. Then there exists an orientation preserving diffeomorphism $f'_Y : \Sigma_{g,n} \to Y$ such that, for each standard loop γ, the image $f'_Y(\gamma) \subset Y$ is homotopic to a loop of length bounded above by $N'(g, n, K)$ relative to $f'_Y(s)$, where $N'(g, n, K)$ depends only on g, n and K.*

Proof. We present $2g + n + [n = 0]$ free loops on $\Sigma_{g,n}$ where $[n = 0] = 1$ if $n = 0$ and $[n = 0] = 0$ if $n \neq 0$, denoted by $\delta_0, \ldots, \delta_{2g+n-1+[n=0]}$, that form a set of closed curves filling up the surface $\Sigma_{g,n}$.

- When $n = 0$, set

$$\begin{aligned}
&\delta_0 = \gamma_{g,2}^{-1},\ \delta_1 = \gamma_{1,1} \cdot \gamma_{g,2}^{-1} \cdot \gamma_{g,1}^{-1} \cdot \gamma_{g,2}, \\
&\delta_{2i} = \gamma_{i,2}^{-1},\ \delta_{2i+1} = \gamma_{i+1,1} \cdot \gamma_{i,2}^{-1} \cdot \gamma_{i,1}^{-1} \cdot \gamma_{i,2}, && \text{for } i = 1, \ldots, g-1, \\
&\delta_{2g} = [\gamma_{1,1}, \gamma_{1,2}].
\end{aligned}$$

- When $g = 0$, set

$$\begin{aligned}
&\delta_0 = {\gamma'}_1^{-1} \cdot {\gamma'}_g^{-1}, \\
&\delta_j = {\gamma'}_{j+1}^{-1} \cdot {\gamma'}_j^{-1}, && \text{for } j = 1, \ldots, n-1.
\end{aligned}$$

- When $g \geq 1$ and $n \geq 1$, set

$$\begin{aligned}
&\delta_0 = \gamma_{1,1} \cdot {\gamma'}_n^{-1}, \\
&\delta_{2i-1} = \gamma_{i,2}^{-1},\ \delta_{2i} = \gamma_{i+1,1} \cdot \gamma_{i,2}^{-1} \cdot \gamma_{i,1}^{-1} \cdot \gamma_{i,2}, && \text{for } i = 1, \ldots, g-1, \\
&\delta_{2g-1} = \gamma_{g,2}^{-1},\ \delta_{2g} = {\gamma'}_1^{-1} \cdot \gamma_{g,2}^{-1} \cdot \gamma_{g,1}^{-1} \cdot \gamma_{g,2}, \\
&\delta_{2g+j} = {\gamma'}_{j+1}^{-1} \cdot {\gamma'}_j^{-1}, && \text{for } j = 1, \ldots, n-1.
\end{aligned}$$

Let Δ_Y be the union of geodesics homotopic to every $\psi \circ f_{X_0}(\delta_i)$. By Wolpert's Lemma,

$$l_Y(\Delta_Y) = \sum_{i=0}^{2g+n-1+[n=0]} L_{\delta_i}([(Y, f_Y)]) \leq e^{2K} \sum_{i=0}^{2g+n-1+[n=0]} L_{\delta_i}([(X_0, f_{X_0})]) \leq 4(2g+n)e^{2K} \cdot N(g, n).$$

Let $t_Y \in \Delta_Y$ be an arbitrary point. Therefore, there exists loops $\widehat{\gamma_{i,1}}, \widehat{\gamma_{i,2}}$ and $\widehat{\gamma'_j}$ at t_Y on Y, for $i = 1, \ldots, g$ and $j = 1, \ldots, n$, that satisfy conditions i, ii and iii, whose lengths are bounded above by

$$4 \cdot 4(2g + n)e^{2K} N(g, n) =: N'.$$

We conclude that there exists a diffeomorphism f'_Y sending $s \in \Sigma_{g,n}$ to t_Y such that, for each standard loop γ, the homotopy class of $f'_Y(\gamma)$ relative to t_Y is represented by a loop of length bounded above by N'. ☐

We aim at addressing the following question : given a hyperbolic surface X of type (g, n), does there exist an orientation preserving diffeomorphism $f'_X : \Sigma_{g,n} \to X$ whose image of each standard loop is homotopic to a loop of short length relative to $f'_X(s)$?

Consider a non-trivial mapping class $[\phi]$ represented by a diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$. Then, the marked hyperbolic surfaces (X, f_X) and $(X, f_X \circ \phi^{-1})$ are the same hyperbolic surface with different markings. Therefore, they possess desired diffeomorphisms for this question simultaneously. Using Mumford's compactness of $\mathcal{M}_{g,n}^{\geq\epsilon}$, we get the following theorem.

Theorem 3.1.11. *There exists a constant $N'' = N''(g, n, \epsilon)$ that depends only on (g, n), ϵ and satisfies the following statement. Given an arbitrary hyperbolic surface Y of type (g, n) with* $\mathrm{sys}(Y) \geq \epsilon$*, there exists an orientation preserving diffeomorphism $f'_Y : \Sigma_{g,n} \to Y$ such that the image $f'_Y(\gamma)$ of each standard loop $\gamma \subset \Sigma_{g,n}$ is homotopic to a loop of length bounded above by N'' relative to $f'_Y(s)$.*

Proof. By Mumford's compactness, the moduli space $\mathcal{M}_{g,n}^{\geq\epsilon}$ is compact and therefore there exists a lift $U \subset \mathcal{T}_{g,n}$ of $\mathcal{M}_{g,n}^{\geq\epsilon}$ containing $[X_0, f_{X_0}]$ whose diameter for the Teichmüller metric is bounded by a constant $D = D(g, n, \epsilon) > 0$ that depends only on (g, n) and ϵ. Suppose that f_Y is an arbitrary marking of Y so that $[(Y, f_Y)] \in \mathcal{T}_{g,n}^{\geq\epsilon}$. Then, there exists an orientation preserving diffeomorphism $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ with $[(Y, f_Y \circ \phi^{-1})] \in U$. Moreover, there exists a K-quasiconformal diffeomorphism $\psi : X_0 \to Y$ homotopic to $(f_Y \circ \phi^{-1}) \circ f_{X_0}^{-1}$, where $e^{2K} \leq D$. Let $f'_Y : \Sigma_{g,n} \to Y$ be the diffeomorphism as in the previous lemma and take $N'' = N'(g, n, K)$. Therefore, for each standard loop γ on $\Sigma_{g,n}$, the image $f'_Y(\gamma)$ is homotopic to a loop of length bounded above by N'' relative to $f'_Y(s)$. □

3.1.5 Bers-Maskit slice

The following proposition is the most technical step in our study on the Parshin-Arakelov finiteness, known as the *irreducibility* of holomorphic curves in the moduli space. It first appeared in [IS88] as part of the proof of the Parshin-Arakelov finiteness. Shiga later formally formulated this result in [Shi97]. For the case of $n = 0$, the irreducibility is a consequence of [DW07, Theorem 5.7] and is also proved using the maximum principle for subharmonic functions [McM00, Theorem 3.1]).

Proposition 3.1.12. *Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Suppose that $F_* : \pi_1(B, t) \to \mathrm{Mod}_h$ is a monodromy homomorphism induced by F with $t \in B$. Then there does not exist a set of non-homotopic disjoint simple closed curves $\{\alpha_1, \ldots, \alpha_{h'}\}$ on Σ_h such that the set of homotopy classes $\{[\alpha_1], \ldots, [\alpha_{h'}]\}$ is preserved by $F_*(g)$ for each $g \in \pi_1(B, t)$.*

This subsection revisits Kleinian groups and the analytic structure of Teichmüller spaces that were first introduced by Bers ([Ber70]) and Maskit ([Mas70]). We include a proof of Proposition 3.1.12 at the end.

Projective linear group. By convention, we use $\hat{\mathbb{C}}$ to denote the Riemann sphere and the upper (resp. lower) half plane is expressed by $\mathcal{U}$ (resp. $\mathcal{L}$) $\subset \mathbb{C} \subset \hat{\mathbb{C}}$, where $\mathbb{H}^2 = \mathcal{U}$. The matrix group $\mathrm{PSL}(2, \mathbb{C})$ acts on $\hat{\mathbb{C}}$ via Möbius transformations. An element $1 \neq g \in \mathrm{PSL}(2, \mathbb{C})$ is called *parabolic* if $\mathrm{tr}^2(g) = 4$, *elliptic* if $\mathrm{tr}^2(g) \in (0, 4) \subset \mathbb{R}$ and *loxodromic* in all other cases. A loxodromic or elliptic element $g \in \mathrm{PSL}(2, \mathbb{C})$ is diagonalizable and conjugate to a transformations $z \to \lambda_g z$ with $|\lambda_g| \geq 1$, where λ_g is called the *multiplier.*

The matrix group $\mathrm{PSL}(2, \mathbb{R})$ is regarded as the largest subgroup of $\mathrm{PSL}(2, \mathbb{C})$ that fixes the upper half plane $\mathcal{U}$. The natural embedding takes hyperbolic, parabolic and elliptic elements into loxodromic, parabolic and elliptic elements, respectively. Therefore, the multiplier λ_g of any hyperbolic element $g \in \mathrm{PSL}(2, \mathbb{R})$ is real and positive such that the translation length of g is exactly $\log \lambda_g$. We always suppose that a Fuchsian group $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ is finitely generated and of the first type.

Let G be an arbitrary subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The *limit set* $\Lambda(G)$ is the closure of the set of fixed points of parabolic and loxodromic elements of G. Besides, $\Lambda(G)$ is also the set of accumulation points of orbits $Gz = \{g(z) \mid g \in G\}$.

Kleinian group. A *Kleinian group G* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ that acts discontinuously on some open subset of $\hat{\mathbb{C}}$. We always suppose that a Kleinian group is finitely generated and *non-elementary,* i.e. $\Lambda(G)$ has more than two points. The subset $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$ is the largest open set on which G acts discontinuously. If there exists a simply-connected component Δ of $\Omega(G)$

that is invariant under G, then G is called a *B-group.* A B-group for which $\Omega(G) = \Delta$ is simply connected is called a *totally degenerate group.*

If G is a B-group with some invariant simply-connected component $\Delta \subset \Omega(G) \subset \hat{\mathbb{C}}$, then there exists a conformal bijection $w_\Delta : \mathcal{U} \to \Delta$ (or $w_\Delta : \mathcal{L} \to \Delta$) so that $w^{-1}Gw \leq \mathrm{PSL}(2, \mathbb{R}) \leq \mathrm{PSL}(2, \mathbb{C})$ is a Fuchsian group isomorphic to G, called the *Fuchsian equivalent* of G. The isomorphism $G \ni g \mapsto w^{-1}gw \in w^{-1}Gw$ is denoted by χ_{w_Δ}. If χ_{w_Δ} sends a parabolic element $g \in G$ to a hyperbolic element, then g is called *accidental parabolic* under the group isomorphism.

Lemma 3.1.13. *Let G be a B-group with exactly one invariant simply-connected component Δ of $\hat{\mathbb{C}} \setminus \Lambda(G)$. If there is no accidental parabolic element under χ_{w_Δ}, then G is totally degenerate.*

Proof. See the corollary to Proposition 7 in [Ber70] or Theorem 4 in [Mas70]. □

Quasi-Fuchsian group. A *quasi-Fuchsian* group $G \leq \mathrm{PSL}(2, \mathbb{C})$ is a Kleinian group such that there exists an invariant (directed) Jordan curve $C \subset \hat{\mathbb{C}}$, which implies a pair of Fuchsian groups $(G^{\mathcal{L}}, G^{\mathcal{U}})$. Suppose that $\Delta^{\mathcal{L}}$, $\Delta^{\mathcal{U}}$ are connected components of $\hat{\mathbb{C}} \setminus C$ and consider the corresponding conformal bijections $w^{\mathcal{L}} : \mathcal{L} \to \Delta^{\mathcal{L}}$, $w^{\mathcal{U}} : \mathcal{U} \to \Delta^{\mathcal{U}}$. Therefore, $(G^{\mathcal{L}}, G^{\mathcal{U}}) = (\chi_{w^{\mathcal{L}}}(\mathcal{U}), \chi_{w^{\mathcal{U}}}(\mathcal{U}))$.

Bounded quadratic differentials. Let $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group. In [Ber70], Bers introduced the finite-dimensional complex Banach space $B_2(\mathcal{L}, \Gamma)$ of *bounded quadratic differentials* for Γ in the lower half plane $\mathcal{L}$, i.e. holomorphic functions $\phi : \mathcal{L} \to \mathbb{C}$ such that $|(\mathrm{Im}(z)^2\phi(z)|$ is bounded and $\phi(g(z))g'(z)^2 = \phi(z)$ for each $g \in \Gamma$.

For each $\phi \in B_2(\mathcal{L}, \Gamma)$, there exists a conformal Schwarzian antiderivative of ϕ, denoted by $w_\phi : \mathcal{L} \to \hat{\mathbb{C}}$. Therefore, the following group homomorphism sends Γ to a Kleinian group Γ_ϕ.

$$\Gamma \ni g \mapsto \Xi_\phi(g) \coloneqq w_\phi g w_\phi^{-1} \in w_\phi \Gamma w_\phi^{-1} \eqqcolon \Gamma_\phi \leq \mathrm{PSL}(2, \mathbb{C}).$$

Lemma 3.1.14 (Lemma 2 in [Ber70])**.** *Let $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group. Then, for any $g \in \Gamma$, the function $\phi \mapsto \mathrm{trace}^2(\Xi_\phi(g))$ is holomorphic defined in $B_2(\mathcal{L}, \Gamma)$.*

We define $\mathcal{T}(\Gamma) \subset B_2(\mathcal{L}, \Gamma)$, now known as the Bers-Maskit slice, as the set of $\phi \in B_2(\mathcal{L}, \Gamma)$ such that w_ϕ admits a quasiconformal extension to all of $\hat{\mathbb{C}}$, denoted by $\hat{w}_\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. For any $\phi \in \mathcal{T}(\Gamma)$, the Kleinian group Γ_ϕ is a quasi-Fuchsian group with invariant Jordan curve $w_\phi(\partial\mathcal{U})$, which implies a pair of Fuchsian groups $(\Gamma_\phi^{\mathcal{L}}, \Gamma_\phi^{\mathcal{U}})$ but one is exactly Γ. By convention, we suppose that $\Gamma = \Gamma_\phi^{\mathcal{L}}$ and call $\Gamma_\phi^{\mathcal{U}}$ the *upper equivalent* of Γ_ϕ. There exists a natural isomorphism between Fuchsian groups $\Gamma = \Gamma_\phi^{\mathcal{L}}$ and $\Gamma_\phi^{\mathcal{U}}$, denoted by $\hat{\Xi}_\phi$, which further induces a quasiconformal map of $\mathcal{U}$.

Lemma 3.1.15. *Given a Fuchsian group Γ, we have the following statements.*

(i) The Bers-Maskit slice $\mathcal{T}(\Gamma)$ is bounded in $B_2(\mathcal{L}, \Gamma)$.
(ii) For any $\phi \in \overline{\mathcal{T}(\Gamma)}$, then $\Xi_\gamma : \Gamma \to \Gamma_\phi$ is an isomorphism to a Kleinian group.
(iii) For any $\phi \in \partial\mathcal{T}(\Gamma)$, then Γ_ϕ is a B-group that contains precisely one invariant component Δ, which is called a boundary group.

Proof. Assertion (i) comes from the famous Nehari–Kraus's theorem. Assertion (ii) comes from Theorem 5 in [Ber70]. Assertion (iii) comes from Theorem 8 in [Ber70] where the invariant component Δ is hence simply-connected and $\chi_w(\Gamma_\phi)$ is a Fuchsian equivalent of the boundary group, for some conformal bijection $w : \mathcal{U} \to \Delta$. □

Bers embedding of Teichmüller space. Suppose that the base point $[(X_0, f_{X_0})] \in \mathcal{T}_h$ has the hyperbolic representative (X_0, f_{X_0}) where X_0 is induced by a Fuchsian group $\Gamma_0 \leq \mathrm{PSL}(2, \mathbb{R})$. Teichmüller space $\mathcal{T}_h$ is interpreted as the space of equivalent classes of quasiconformal maps of $\mathcal{U} = \mathbb{H}^2$ compatible with Γ_0, where two quasiconformal maps are equivalent if they coincide on the boundary up to pre-composition by Möbius transformations.

Given a quasiconformal map $w : \mathcal{U} \to \mathcal{U}$, Teichmüller space $\mathcal{T}_h$ can be identified with $\mathcal{T}(\Gamma_0)$ by taking the Schwarzian derivative ϕ of a natural extension $\hat{w}$ of w. In fact, the quasiconformal map induced by $\hat{\Xi}_\phi$ is equivalent to w. The analytic structure on $\mathcal{T}_h = \mathcal{T}(\Gamma_0)$ comes from the Banach space $B_2(\mathcal{L}, \Gamma_0)$ and it is a bounded domain of a finite-dimensional complex Banach space.

Lemma 3.1.16 (Abikoff's lemma)**.** *Take $\phi \in \partial\mathcal{T}(\Gamma_0)$ such that the boundary group Γ_ϕ is totally degenerate and let $g \in \Gamma_\phi$ be a loxodromic element. If there exist a sequence $\{\phi_n\}$ in $\mathcal{T}(\Gamma_0)$ that converges to ϕ as $n \to \infty$ and some hyperbolic $g_0 \in \Gamma_0$ such that $g_n \coloneqq \Xi_{\phi_n}(g_0) \in \Gamma_{\phi_n} \leq \mathrm{PSL}(2, \mathbb{C})$ converges to g as $n \to \infty$, then $L_{f_{X_0}^{-1}(\gamma_0)}(\phi_n)$ is unbounded, where $\gamma_0 \subset X_0$ is an arbitrary closed curve corresponding to g_0*

Proof. See Theorem 2 in [Abi76] and use the fact that $L_{f_{X_0}^{-1}(\gamma_0)}(\phi_n) = \lambda_{\hat{\Xi}_{\phi_n}(g_0)}$. $\square$

Now we prove Proposition 3.1.12.

Proof of Proposition 3.1.12. Consider the non-constant holomorphic map $F : B \to \mathcal{M}_h$. Let $\Delta \subset \mathbb{C}$ be the unit disc and suppose that $\iota : \Delta \to B$ is a universal covering whose cover transformation group is $\pi_1(B, t)$. Therefore, F induces the non-constant holomorphic map of the unit disc Δ to $\mathcal{T}_h$, denoted by $\widetilde{F} : \Delta \to \mathcal{T}_h$.

For each $g \in \pi_1(B, t)$ and α_i, we have

$$L_{\alpha_i}(\widetilde{F}(g(0))) = L_{F_*(g)(\alpha_i)}(\widetilde{F}(0)) \leq \max\{L_{\alpha_j}(\widetilde{F}(0)) \mid j = 1, \dots, h'\}.$$

Since $\widetilde{F}(\Delta) \subset \mathcal{T}_h = \mathcal{T}(\Gamma_0) \subset B_2(\mathcal{L}, \Gamma_0)$ is bounded, Fatou's theorem concerning holomorphic maps on the unit disc states that for almost all $\zeta \in \partial\Delta$, the holomorphic map $\widetilde{F}$ converges to some point $q \in \overline{\mathcal{T}(\Gamma_0)}$ non-tangentially. Since the cover transformation group $\pi_1(B, t)$ is of divergence type, there exists a sequence of transformations $g_{\zeta,j} \in \pi_1(B, t)$ such that $g_{\zeta,j}(z_0)$ converges to ζ non-tangentially and $\widetilde{F}(g_{\zeta,j}(z_0))$ converges to q as $j \to \infty$, for every $z_0 \in \Delta$ (see Theorem XI 20 in [Tsu75]).

We claim that the image limit $q \in \partial\mathcal{T}(\Gamma_0)$. Indeed, the monodromy $F_*(g_{\zeta,j}) \in \mathrm{Mod}_h$ along $g_{\zeta,j}$ is a transformation on $\mathcal{T}_h = \mathcal{T}(\Gamma_0)$, which is isometric and holomorphic. These transformations converge to a holomorphic map $A : \mathcal{T}_h \to \overline{\mathcal{T}_h}$. Since z_0 is arbitrary, if $q \in \mathcal{T}_h$ then A, as the limit of a sequence of isometries, cannot be constant, which is a contradiction.

By Lemma 3.1.13 and Lemma 3.1.15 - (iii), if a boundary group Γ_q, for $q \in \partial\mathcal{T}(\Gamma_0)$, has no accidenal parabolic element, then Γ_q is totally degenerate. However, by Lemma 3.1.14 and Lemma 3.1.15 - (i), the bounded holomorphic map

$$\Delta \ni z \mapsto \mathrm{trace}^2(\Xi_{\widetilde{F}(z)}(g))$$

cannot have non-tangential limits that is equal to 4 almost everywhere on $\partial\mathcal{T}(\Gamma_0)$, for each hyperbolic transformation $g \in \Gamma_0$. Besides, Γ_0 is finitely generated and therefore countable. Hence, for almost all possible non-tangential image limits $q \in \partial\mathcal{T}(\Gamma_0)$ of $\widetilde{F}$, boundary groups Γ_q are totally degenerate.

Let $\zeta \in \partial\Delta$ be a boundary point and $g_{\zeta,j} \in \pi_1(B, t)$ be a sequence of transformations such that $g_{\zeta,j}(0) =: z_j \in \Delta$ converges to $\zeta \in \partial\Delta$ non-tangentially, $\widetilde{F}(z_j)$ converges to $q \in \partial\mathcal{T}(\Gamma_0)$ as $j \to \infty$, the boundary group $\Gamma_q \leq \mathrm{PSL}(2, \mathbb{C})$ is totally degenerate and has no accidental parabolic element under $\Xi_q^{-1} : \Gamma_q \to \Gamma_0$. Let $g_0 \in \Gamma_0$ be hyperbolic and corresponding to $f_{X_0}(\alpha_i) \subset X_0$. By Lemma 3.1.15 - (ii), transformations $\Xi_{\widetilde{F}(z_j)}(g_0)$ must converge to a loxodromic element $\Xi_q(g_0)$. By Lemma 3.1.16, we get

$$L_{\alpha_i}(\widetilde{F}(z_j)) \to \infty \text{ as } j \to \infty,$$

a contradiction. $\square$

3.2 Uniform boundedness for Parshin-Arakelov finiteness

Parshin-Arakelov finiteness investigates non-isotrivial holomorphic families of Riemann surfaces over a Riemann surface of finite type. Suppose that B is an oriented hyperbolic surface of type (g, n). A holomorphic family C/B that comes from a holomorphic map $F : B \to \mathcal{M}_h$ is non-isotrivial if and only of F is non-constant. Parshin-Arakelov finiteness claims that there are only finitely many non-isotrivial non-isomorphic families of closed Riemann surfaces of genus h over B. This also means that there are only finitely many non-constant holomorphic maps $F : B \to \mathcal{M}_h$.

Fix $\epsilon > 0$. In this section, however, we investigate a non-constant holomorphic map $F : B \to \mathcal{M}_h$ where B is an arbitrary, not fixed, oriented hyperbolic surface of type (g, n) with $\mathrm{sys}(B) \geq \epsilon$. In particular, we prove Theorem G from the introduction.

Theorem 3.2.1. *There exist a constant $N'' = N''(g, n, \epsilon)$ depending only on (g, n), ϵ and a compact subset $\mathcal{K}' = \mathcal{K}'(g, n, h, \epsilon) \subset \mathcal{M}_h$ depending only on $(g, n), h, \epsilon$ that satisfy the following statement. Let B be an oriented hyperbolic surface of type (g, n) such that $\mathrm{sys}(B) \geq \epsilon$ and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Then, there exists an orientation preserving diffeomorphism $f'_B : \Sigma_{g,n} \to B$ such that*

(i) f'_B sends the base point $s \in \Sigma_{g,n}$ to $t'_B \in B$ such that $F(t'_B) \in \mathcal{K}'$;

(ii) the image $f'_B(\gamma) \subset B$ of each standard loop $\gamma \subset \Sigma_{g,n}$ is homotopic to a loop relative to t'_B of length bounded above by N''.

Theorem 3.2.1 is a continuation of Theorem 3.1.11, in which the constant $N''(g, n, \epsilon)$ and the orientation preserving diffeomorphism f'_B are inherited. We start with a well-known lemma on closed hyperbolic surface, based on which we then prove that $\mathrm{sys}(F(t'_B))$ is bounded. The proof of Theorem 3.2.1 proceeds similarly to the proof of [McM00, Theorem 3.1].

Lemma 3.2.2 (Corollary 13.7 in [FM11]). *There exists a constant $\tau_h > 0$ depending only on h such that on any closed oriented hyperbolic surface X of genus h, if $\{\alpha_1, \dots, \alpha_{h'}\}$ is the set of closed geodesics of length smaller than τ_h, then $\iota(\alpha_i, \alpha_j) = 0$ and $h' \leq 3h - 3$.*

Proof of Theorem 3.2.1. We aim to show that $\mathrm{sys}(F(t'_B)) \geq \tau_h / N''^{3h-3}$ and therefore take $\mathcal{K}' = \mathcal{M}_h^{\geq \tau_h / N''^{3h-3}}$, which is a compact subset by Mumford's compactness. Suppose that $F(t'_B)$ has a hyperbolic representative (X, f_X) and assume that $\mathrm{sys}(X) < \tau_h / N''^{3h-3}$. By Lemma 3.2.2, let $\alpha_1, \dots, \alpha_{h'}$ be closed geodesics on X of length smaller than τ_h, with $h' \leq 3h - 3$.

By Wolpert's Lemma, images under f_X^{-1} of the shortest several of $\alpha_1, \dots, \alpha_{h'}$ form a set of homotopy classes on Σ_h that is preserved by $F_*(g) \in \mathrm{Mod}_h$ for each $g \in \pi_1(B, t'_B)$. This contradicts with Proposition 3.1.12. ☐

Now, we provide the uniform boundedness for Parshin-Arakelov finiteness.

Theorem 3.2.3. *The subset*

$$\left\{ \boldsymbol{MO}(F) \,\middle|\, \begin{array}{l} B \text{ is an oriented hyperbolic surface of type } (g,n) \text{ such that } \mathrm{sys}(B) \geq \epsilon \\ F : B \to \mathcal{M}_h \text{ is a non-constant holomorphic map} \end{array} \right\} \in M_{g,n,h}$$

is finite, where the finiteness depends only on g, n, h and ϵ.

Proof. Let $\widetilde{F} : \mathbb{H}^2 \to \mathcal{T}_h$ be the lift of F. Consider a monodroy homomorphism $F_* = F_\Gamma \circ \rho_{t,\widetilde{t}} \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$. By Theorem 3.1.11 and Theorem 3.2.1, there exists an orientation preserving diffeomorphism $f'_B : \Sigma_{g,n} \to B$ such that $f'_B(s) = t'_B$ and, for each standand loop $\gamma \subset \Sigma_{g,n}$, there exists a loop $\gamma_B \subset B$ homotopic to $f'_B(\gamma)$ relative to t'_B such that $l_B(\gamma_B) \leq N'' = N''(g, n, \epsilon)$. Besides, we have a compact subset $\mathcal{K}' = \mathcal{K}'(g, n, h, \epsilon) \subset \mathcal{M}_h$ such that $F(t'_B) \in \mathcal{K}'$.

When $t = t'_B$, it suffices to show that there are only finitely many possibilities of $F_*([\gamma_B]) \in \mathrm{Mod}_h$ for each standard loop γ. Since $\mathcal{K}'$ is compact, it has a bounded lift $\widetilde{\mathcal{K}'} \subset \mathcal{T}_h$ containing $\widetilde{F}(\widetilde{t})$. For each standard loop $\gamma \subset \Gamma_{g,n}$, by Proposition 3.1.2, the mapping class $g := F_*([\gamma_B]) \in \mathrm{Mod}_h$ has to be such that $d_{\mathcal{T}}(\widetilde{F}(\widetilde{t}), g \cdot \widetilde{F}(\widetilde{t})) \leq N''/2$. As the Teichmüller metric is proper, the subset $\mathrm{B}(\widetilde{\mathcal{K}'}, N''/2)$ is again compact. Since the mapping class group Mod_h acts properly discontinuously on $\mathcal{T}_h$, there are only finitely many choices of g.

In general, there exists an orientation preserving diffeomorphism $f : \Sigma_{g,n} \to B$ such that $f(s) = t$ and f is homotopic to f'_B. Let $H : [0, 1] \times \Sigma_{g,n} \to B$ be the homotopy such that $H(0, \cdot) = f(\cdot)$ and $H(1, \cdot) = f'_B(\cdot)$. The path $H(\cdot, s)$ joining t to t'_B induces a new monodromy homomorphism $F'_* \in \mathrm{Hom}(\pi_1(B, t'_B), \mathrm{Mod}_h)$ such that $F_*([f(\gamma)]) = F'_*([f'_B(\gamma)])$ for each standard loop γ. Hence, there are only finitely many possibilities of $F_* \circ f_*$. ☐

3.3 Quasi-isometric rigidity of holomorphic curves

Suppose that B is a hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ is a holomorphic map. In this section, we introduce the rigidity result, which claims that the holomorphic curve $F(B)$ is very similar to a Teichmüller curve.

3.3.1 From cusp region to end of moduli space

The moduli space $\mathcal{M}_h$ has only one end, meaning that for any compact set, there is exactly one unbounded component of the complement. In this subsection, we consider a hyperbolic cusp region U, i.e. the neighbourhood of a cusp bounded by a horocycle of length 2. Then we investigate a non-constant map $F : U \to \mathcal{M}_h$ that is distance-decreasing for 1/2 of the hyperbolic distance d_U on U and the Teichmüller distance $d_{\mathcal{M}}$ on $\mathcal{M}_h$.

The Dehn twist along a closed curve $\alpha \subset \Sigma_h$, denoted by τ_α, is a diffeomorphism of Σ_h and represents a mapping class $T_\alpha \in \mathrm{Mod}_h$. Let $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_m\}$ be a multi-curve, then a multi-twist along $\boldsymbol{\alpha}$ is a product of the form $T = T_{\alpha_1}^{r_1} \circ \dots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ with each $r_i \in \mathbb{Z} \setminus \{0\}$. In particular, a power of positive or negative Dehn twist is a multi-twist in our discussion.

Lemma 3.3.1. *Let $\widetilde{\gamma_1}, \widetilde{\gamma_2} \subset \mathbb{H}^2$ be disjoint geodesics and $k \geq 3$ be an integer. Suppose that ϕ is a hyperbolic isometry along $\widetilde{\gamma_1}$ whose translation length is equal to l such that $\widetilde{\gamma_2} \cap \phi(\widetilde{\gamma_2}) = \emptyset$. Let $\widetilde{p_1}, \widetilde{p_2} \in \mathbb{H}^2$ be arbitrary points partitioned by both $\widetilde{\gamma_2}$ and $\phi^k(\widetilde{\gamma_2})$. Then $d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq l$.*

Proof. Take the (unique) geodesic segment β_i perpendicular to both $\widetilde{\gamma_1}$ and $\phi^i(\widetilde{\gamma_2})$, for $i = 1, 2$. The hyperbolic plane is separated by $\widetilde{\gamma_1}$, $\widetilde{\gamma_2}$, $\phi(\widetilde{\gamma_2})$, $\phi^2(\widetilde{\gamma_2})$, $\phi^k(\widetilde{\gamma_2})$ and β_1, β_2 into 8 pieces. Therefore, the geodesic segment joining $\widetilde{p_1}$ to $\widetilde{p_2}$ goes cross β_1 and β_2. Hence the distance between $\widetilde{p_1}$ and $\widetilde{p_2}$ is at least the distance between β_1 and β_2, which is equal to l. □

Lemma 3.3.2 (Lemma 4.2 in [Iva92], cf. Exposé 6, Section VII in [FLP12]). *Let $\tau = \tau_{\alpha_1}^{r_1} \circ \cdots \circ \tau_{\alpha_m}^{r_m}$ be a multi-twist diffeomorphism along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$. Then for all closed curves $\gamma_1, \gamma_2 \subset \Sigma_h$, we have*

$$\iota(\tau(\gamma_1), \gamma_2) \geq \sum_{i=1}^{m} (|r_i| - 2)\, \iota(\gamma_1, \alpha_i)\, \iota(\gamma_2, \alpha_i) - \iota(\gamma_1, \gamma_2).$$

In particular, for any multi-twist diffeomorphism τ along $\boldsymbol{\alpha}$ and closed curve γ intersecting $\boldsymbol{\alpha}$ at least once, we have $\iota(\tau^3(\gamma), \gamma) \geq \sum_{i=1}^{m} (|3r_i| - 2)\, \iota(\gamma, \alpha_i)^2 \geq 1$. The following lemma is inspired by this observation.

Lemma 3.3.3. *Let $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ be a multi-twist along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ and $\gamma \subset \Sigma_h$ be a simple closed curve such that $\iota(\boldsymbol{\alpha}, \gamma) \coloneqq \sum_i |r_m|\, \iota(\alpha, \gamma) \geq 1$. Then, given $[(X, f_X)] \in \mathcal{T}_h$, we have*

$$L_\gamma([(X, f_X)]) + L_\gamma(T^4 \cdot [(X, f_X)]) \geq \frac{1}{3} L_{\alpha_i}([(X, f_X)])$$

for all $i = 1, \ldots, m$.

Proof. Without loss of generality, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$. Suppose that T is represented by a multi-twist diffeomorphism τ. We have

$$L_\gamma(T^4 \cdot [(X, f_X)]) = L_{\tau^{-4}(\gamma)}([(X, f_X)])$$

since $T \cdot [(X, f_X)] = [(X, f_X \circ \tau^{-1})]$.

Consider a universal covering $\iota : \mathbb{H}^2 \to X$ such that the horizontal geodesic $\widetilde{\gamma_1} \subset \mathbb{H}^2$ is a lift of the (unique) geodesic homotopic to $f_X(\gamma) \subset X$ (see Figure 3.2). We suppose that $\widetilde{\gamma_1}$ is oriented toward the left. Proceeding from $0 \in \mathbb{H}^2$, the first lift of some $f_X(\alpha_u)$ that intersects $\widetilde{\gamma_1}$ is denoted by $\widetilde{\alpha_1}$ and the second lift of some $f_X(\alpha_v)$ is denoted by $\widetilde{\alpha_2}$. Going along the opposite direction, the first lift of some $f_X(\alpha_x)$ is denoted by $\widetilde{\alpha_{-1}}$ and the second lift of some $f_X(\alpha_y)$ is denoted by $\widetilde{\alpha_{-2}}$. Suppose that $\widetilde{\alpha_1}$ and $\widetilde{\alpha_{-1}}$ are oriented upward.

Fix $i = 1, \ldots, m$. Without loss of generality, we further assume that $u = i$.

The closed geodesic homotopic to $f_X(\alpha_u) \subset X$ is interpreted by a hyperbolic isometry $\phi_1 \in \mathrm{Isom}(\mathbb{H}^2)$ of which $\widetilde{\alpha_1}$ is the axis, whose translation length is $L_{\alpha_u}([(X, f_X)])$. Therefore, the twist $T_{\alpha_u}^{r_u}$ is interpreted by $\phi_1^{r_u}$ acting on the left. Similarly, the closed geodesic homotopic to $f_X(\alpha_x) \subset X$ is interpreted by a hyperbolic isometry $\phi_{-1} \in \mathrm{Isom}(\mathbb{H}^2)$ of which $\widetilde{\alpha_{-1}}$ is the axis, whose translation length is $L_{\alpha_x}([(X, f_X)])$. Therefore, the twist $T_{\alpha_x}^{-r_x}$ is interpreted by $\phi_{-1}^{-r_x}$ acting on the right.

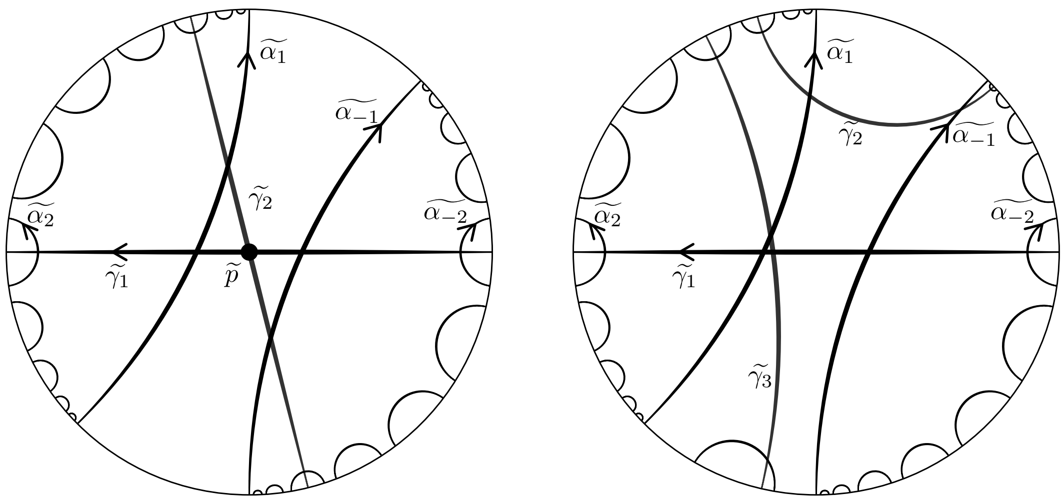


FIGURE 3.2 – Lifts of γ and $T^4(\gamma)$ given a multi-twist T along $\boldsymbol{\alpha}$.

Suppose that the twists within T along α_u and α_x have the same direction. Without lose of generality, we assume that $r_u > 0$ and $r_x > 0$. There exists a lift $\widetilde{\gamma_2}$ of the (unique) geodesic homotopic to $f_X(\tau^{-4}(\gamma)) \subset X$ which connects two boundary points partitioned by both $\phi_1^{4r_u}(\widetilde{\alpha_2})$ and $\phi_{-1}^{-4r_x}(\widetilde{\alpha_{-2}})$. Let $\widetilde{p} \in \mathbb{H}^2$ be the intersection $\widetilde{\gamma_1} \cap \widetilde{\gamma_2}$. Let ψ_1, ψ_2 be hyperbolic isometries along $\widetilde{\gamma_1}$, $\widetilde{\gamma_2}$ corresponding to $f_X(\gamma)$, $f_X(\tau^{-4}(\gamma))$ respectively. Therefore,

$$\begin{aligned}3\Big(L_\gamma([(X, f_X)]) + L_\gamma(T^4 \cdot [(X, f_X)])\Big) &= 3\Big(L_\gamma([(X, f_X)]) + L_{\tau^{-4}(\gamma)}([(X, f_X)])\Big)\\ &= d_{\mathbb{H}^2}(\widetilde{p}, \psi_1^3(\widetilde{p})) + d_{\mathbb{H}^2}(\widetilde{p}, \psi_2^3(\widetilde{p})).\end{aligned}$$

Meanwhile, $\psi_1^3(\widetilde{p})$ and $\psi_2^3(\widetilde{p})$ are partitioned by both $\widetilde{\alpha_2}$ and $\phi_1^{4r_u}(\widetilde{\alpha_2})$. The desired inequality follows from Lemma 3.3.1.

Suppose that twists within T along α_u and α_x have different directions. Without loss of generality, we assume that $r_u > 0$ and $r_x < 0$. In this case, we have $\iota(\boldsymbol{\alpha}, \gamma) \geq 2$. Take the lift $\widetilde{\gamma_2}$ of the geodesic homotopic to $f_X(\tau^{-4}(\gamma)) \subset X$ which connects two boundary points partitioned by both $\phi_1^{4r_u}(\widetilde{\alpha_2})$ and $\phi_{-1}^{-4r_x}(\widetilde{\alpha_{-2}})$. Note that $\widetilde{\gamma_2}$ does not intersect $\widetilde{\gamma_1}$. The geodesic $\widetilde{\gamma_3} = \phi_1^{-1}(\widetilde{\gamma_2})$ is another lift of the geodesic homotopic to $f_X(\tau^{-4}(\gamma))$, which connects two boundary points that are partitioned by $\phi_1^{4r_u-1}(\widetilde{\alpha_2})$ and $\phi_1^{-1}(\widetilde{\alpha_{-1}})$. Hence $\widetilde{\gamma_3}$ intersects $\widetilde{\gamma_1}$, where the intersection is denoted by $\widetilde{p}$. Let ψ_1, ψ_3 be hyperbolic isometries along $\widetilde{\gamma_1}$, $\widetilde{\gamma_3}$ corresponding to $f_X(\gamma)$, $f_X(\tau^{-4}(\gamma))$ respectively. Therefore, $\psi_1^3(\widetilde{p})$ and $\psi_3^3(\widetilde{p})$ are partitioned by both $\widetilde{\alpha_2}$ and $\phi_1^{4r_u-1}(\widetilde{\alpha_2})$. Again, the desired inequality follows from Lemma 3.3.1. $\square$

The multi-twist of a hyperbolic surface that provides a very slight deformation of the hyperbolic structure should be along a set of very short closed geodesics. We formulate this property in Proposition 3.3.4 and Proposition 3.3.6, which should be well-known.

Proposition 3.3.4. *Given $\mu \in \mathbb{Z}_{>0}$, there exists a constant $l_{max} = l_{max}(h, \mu)$ that depends only on h and μ such that, for any multi-twist $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ and $[(X, f_X)] \in \mathcal{T}_h$, if*

$$d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)]) \leq 2\mu$$

then

$$L_{\alpha_i}([(X, f_X)]) \leq l_{max}$$

for all $i = 1, \ldots, m$.

Proof. Without loss of generality, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$. There exists a geodesic pants decomposition $\mathcal{P}_X = \{\gamma_i\}$ of X with each $l_X(\gamma_i)$ bounded above by Bers' constant (cf. Theoem 12.8 in [FM11]). More precisely, $l_X(\gamma_i) \leq 21(h-1)$ for each $\gamma_i \in \mathcal{P}_X$. Suppose that $\alpha_{X,i} \subset X$ is the (unique) geodesic homotopic to $f_X(\alpha_i)$, for each $\alpha_i \in \boldsymbol{\alpha}$, and set $\boldsymbol{\alpha}_X = \{\alpha_{X,i} \mid i = 1, \ldots, m\}$. There are two cases to consider for all $i = 1, \ldots, m$.
Case 1 : $\alpha_{X,i} \in \mathcal{P}_X$**.** Then, $L_{\alpha_i}([(X, f_X)]) = l_X(\alpha_{X,i}) \leq 21(h-1)$.
Case 2 : $\alpha_{X,i} \notin \mathcal{P}_X$**.** Then, there exists a simple closed curve $\gamma \subset \Sigma_h$ such that $\iota(\boldsymbol{\alpha}, \gamma) \geq 1$ and $L_\gamma([(X, f_X)]) \leq 21(h-1)$. By Wolpert's Lemma,

$$L_\gamma(T^4 \cdot [(X, f_X)]) \leq \exp\{2 \cdot d_{\mathcal{T}}([(X, f_X)], T^4 \cdot [(X, f_X)])\} \cdot L_\gamma([(X, f_X)]) \leq e^{16\mu} \cdot L_\gamma([(X, f_X)]).$$

Hence, by Lemma 3.3.3, we get

$$\begin{aligned}L_{\alpha_i}([(X, f_X)]) &\leq 3\Big(L_\gamma([(X, f_X)]) + L_\gamma(T^4 \cdot [(X, f_X)])\Big)\\ &\leq 3L_\gamma([(X, f_X)])(1 + e^{16\mu}) \leq 63(h-1)(1 + e^{16\mu}) =: l_{\max}(h, \mu).\end{aligned}$$

$\square$

We shall consider Fenchel-Nielsen coordinates for $\mathcal{T}_h$ associated with a pants decomposition $\mathcal{P}$ of Σ_h. Let $\mathcal{P} = \{C_i\}$ be a set of closed curves on Σ_h. The *length parameter* of C_i is denoted by $l_i = L_{C_i}([(X, f_X)])$. However, the *twist parameter* θ_i is chosen to be proportional along C_i so that a positive Dehn twist along C_i changes the twist parameter by adding 2π. For $[(X, f_X)], [(X', f_{X'})] \in \mathcal{T}_h$, the *Fenchel-Neilsen distance* with respect to $\mathcal{P}$ is defined by

$$d_{\mathrm{FN},\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) = \sup_i \max\{|\log l_i - \log l_i'|, |l_i\theta_i - l_i'\theta_i'|\}$$

where $[(X, f_X)], [(X', f_{X'})]$ have Fenchel-Nielsen coordinates $c = (l_i, \theta_i)_i$ and $c' = (l_i', \theta_i')_i$ respectively. Fenchel-Nielsen distance is introduced and investigated in [ALPSS11]. In fact, we have the following quasi-isometric relation between Fenchel-Nielsen distance and Teichmüller distance.

Proposition 3.3.5. *Let $[(X, f_X)]$, $[(X', f_{X'})] \in \mathcal{T}_h$ be arbitrary and $\mathcal{P} = \{C_i\}$ be a pants decomposition of Σ_h such that $[(X, f_X)]$, $[(X', f_{X'})]$ have Fenchel-Nielsen coordinates $c = (l_i, \theta_i)_i$, $c' = (l'_i, \theta'_i)_i$ respectively. Suppose that, for some constants $N_1, N_2 > 0$, we have $l_i, l'_i \leq N_1$ for all $i = 1, 2, \ldots$ and $d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \leq N_2$. Then,*

$$d_{\mathcal{T}}([(X, f_X)], [(X', f_{X'})]) \leq d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \cdot N_3(N_1, N_2)$$

and

$$d_{FN,\mathcal{P}}([(X, f_X)], [(X', f_{X'})]) \leq d_{\mathcal{T}}([(X, f_X)], [(X', f_{X'})]) \cdot N_4(N_1)$$

where the constant $N_3(N_1, N_2)$ depends on N_1 and N_2, the constant $N_4(N_1)$ depends only on N_1.

Proof. The first inequality comes from Proposition 8.4 in [ALPSS11] and the second inequality comes from Corollary 8.8 in [ALPSS11]). □

We improve Proposition 3.3.4 using Proposition 3.3.5.

Proposition 3.3.6. *Given $\mu \in \mathbb{Z}_{>0}$, there exists a constant $K_1 = K_1(h, \mu)$ that depends only on h and μ such that, for any multi-twist $T = T_{\alpha_1}^{r_1} \circ \cdots \circ T_{\alpha_m}^{r_m} \in \mathrm{Mod}_h$ along $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_m\}$ and $[(X, f_X)] \in \mathcal{T}_h$, if*

$$d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)]) \leq 2\mu$$

then

$$L_{\alpha_i}([(X, f_X)]) \leq K_1 d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])$$

for all $i = 1, \ldots, m$.

Proof. As in Proposition 3.3.4, we assume that (X, f_X) is a hyperbolic representative of the given $[(X, f_X)]$ and let $\boldsymbol{\alpha}_X$ be the set of geodesics on X homotopic to $f_X(\alpha_i)$ for every $\alpha_i \in \boldsymbol{\alpha}$. Therefore, there exists a(nother) geodesic pants decomposition $\mathcal{P}_X = \{\gamma_i\}$ of X such that

- $\boldsymbol{\alpha}_X \subseteq \mathcal{P}_X$;
- $l_X(\gamma) \leq \mathrm{Bers}(\boldsymbol{\alpha}_X)$ for each $\gamma \in \mathcal{P}_X$, where $\mathrm{Bers}(\boldsymbol{\alpha}_X)$ is a variation of Bers' constant that depends only on h and lengths of every geodesics in $\boldsymbol{\alpha}_X$, therefore depends only on h and μ.

Set $C_i = f_X^{-1}(\gamma_i)$ and then $\mathcal{P} = \{C_i\}$ is a pants decomposition of Σ such that $L_{C_i}([(X, f_X)]) \leq \mathrm{Bers}(\boldsymbol{\alpha}_X)$, for each C_i. By Wolpert's Lemma, we further have

$$L_{C_i}(T \cdot [(X, f_X)]) \leq \exp\{2 \cdot d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])\} \cdot L_{C_i}([X, f_X]) \leq e^{4\mu}\,\mathrm{Bers}(\boldsymbol{\alpha}_X).$$

Therefore, by Proposition 3.3.5, there exists a constant K_1 depending only on $\mathrm{Bers}(\boldsymbol{\alpha}_X)$ and $e^{4\mu}\,\mathrm{Bers}(\boldsymbol{\alpha}_X)$ such that

$$L_{\alpha_i}([(X, f_X)]) \leq \frac{1}{2\pi} d_{\mathrm{FN},\mathcal{P}}([(X, f_X)], T \cdot [(X, f_X)]) \leq K_1 d_{\mathcal{T}}([(X, f_X)], T \cdot [(X, f_X)])$$

for all $i = 1, \ldots, m$. □

From now on, we consider a hyperbolic cusp region U and a non-constant map $F : U \to \mathcal{M}_h$ that is distance-decreasing for $(1/2)d_U$ on U and $d_{\mathcal{M}}$ on $\mathcal{M}_h$.

Suppose that $U = \langle g \rangle \backslash \mathcal{B}$ with $\mathcal{B} \subset \mathbb{H}^2$ a horoball and $g \in \mathrm{Aut}(\mathbb{H}^2)$ a parabolic isometry. Therefore, the map F can be lifted to a map $\widetilde{F} : \mathcal{B} \to \mathcal{T}_h$. Let $\phi \in \mathrm{Mod}_h$ be such that, given a generating loop $\gamma \subset U$ based at $p \in U$ and a lift $\widetilde{p} \in \mathcal{B}$ of p, one can lift $F(\gamma)$ to a path joining $\widetilde{F}(\widetilde{p})$ to $\phi \cdot \widetilde{F}(\widetilde{p})$. This mapping class ϕ should satisfy the inequality $\epsilon/2 \geq d_{\mathcal{T}}(\widetilde{F}(\widetilde{p}), \phi \cdot \widetilde{F}(\widetilde{p}))$, where $0 < \epsilon \leq 2$ and the horocycle $H_\epsilon \subset U$ of length ϵ contains p. Such a mapping class is called the monodromy of F.

We suppose that a power ϕ^μ is exactly a multi-twist T along $\boldsymbol{\alpha}$. Each point $p \in U$ determines the unique horocycle $H_\varepsilon \subset U$ such that $p \in H_\varepsilon$. Each lift $\widetilde{p} \in \mathcal{B}$ of p determines the geodesic length $l = L_{\alpha_1}(\widetilde{F}(\widetilde{p}))$. We associate the length amount ϵ and the length amount l to show that F is a quasi-isometric embedding.

Theorem 3.3.7. *Given $\epsilon > 0$, there exists $K_2 = K_2(h, \mu, \epsilon)$ that depends only on h, μ, ϵ and satisfies the following statement. Suppose that $\mathrm{sys}(F(p_{max})) \geq \epsilon$ for some $p_{max} \in \partial U$. Then, we have*

$$\frac{1}{2} d_U(p_1, p_2) \geq d_{\mathcal{M}}(F(p_1), F(p_2)) \geq \frac{1}{2} d_U(p_1, p_2) - K_2$$

for any pair of points (p_1, p_2) in U.

Proof. Let $p_{\max} \in \partial U$ and $p \in H_\varepsilon \subset U$ be arbitrary with $\varepsilon \leq 2$. Take a lift $\widetilde{p_{\max}} \in \mathcal{B}$ of $p_{\max}$ and a lift $\widetilde{p} \in \mathcal{B}$ of p such that $d_{\mathcal{M}}(F(p_{\max}), F(p)) = d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_{\max}}), \widetilde{F}(\widetilde{p}))$. For convenience, we set $q_{\max} = F(p_{\max})$, $q = F(p)$, $\widetilde{q_{\max}} = \widetilde{F}(\widetilde{p_{\max}})$ and $\widetilde{q} = \widetilde{F}(\widetilde{p})$. By Wolpert's Lemma, Proposition 3.3.6 and the triangle inequality in (U, d_U), we have

$$\begin{aligned}
d_{\mathcal{M}}(q, q_{\max}) = d_{\mathcal{T}}(\widetilde{q}, \widetilde{q_{\max}}) &\geq \frac{1}{2} \log \frac{L_{\alpha_1}(\widetilde{q_{\max}})}{L_{\alpha_1}(\widetilde{q})} \\
&\geq \frac{1}{2} \log \frac{\mathrm{sys}(\widetilde{q_{\max}})}{K_1(h,\mu) d_{\mathcal{T}}(\widetilde{q}, T \cdot \widetilde{q})} \geq \frac{1}{2} \log \frac{\mathrm{sys}(q_{\max})}{K_1(h,\mu)\mu\epsilon/2} = \frac{1}{2}\Big\{ \log \frac{\mathrm{sys}(q_{\max})}{K_1(h,\mu)\mu} + \log \frac{2}{\epsilon} \Big\} \\
&\geq \frac{1}{2} d_U(p, p_{\max}) - K_2'
\end{aligned}$$

where $K_2' = 1 - (1/2)\log \mathrm{sys}(q_{\max}) + (1/2) \log K_1(h,\mu)\mu$.

In general, let $p_1, p_2 \in U$ be arbitrary. Set $q_1 = F(p_1)$, $q_2 = F(p_2)$ and take the corresponding horocycles $H_{\varepsilon_1} \ni p_1$, $H_{\varepsilon_2} \ni p_2$. Using the above inequality and triangle inequalities in both (U, d_U) and $(\mathcal{M}_h, d_{\mathcal{M}})$, we conclude that

$$\begin{aligned}
\frac{1}{2} d_U(p_1, p_2) &\geq d_{\mathcal{M}}(q_1, q_2) \\
&\geq |d_{\mathcal{M}}(q_1, q_{\max}) - d_{\mathcal{M}}(q_2, q_{\max})| \\
&\geq \frac{1}{2} |d_U(p_1, p_{\max}) - d_U(p_2, p_{\max})| - K_2' \\
&\geq \frac{1}{2} \max \Big\{ \Big(\log \frac{2}{\varepsilon_1} - 2 \Big) - \Big(\log \frac{2}{\varepsilon_2} + 2 \Big), \Big(\log \frac{2}{\varepsilon_2} - 2 \Big) - \Big(\log \frac{2}{\varepsilon_1} + 2 \Big) \Big\} - K_2' \\
&= \frac{1}{2} \max \Big\{ \log \frac{\varepsilon_1}{\varepsilon_2}, \log \frac{\varepsilon_2}{\varepsilon_1} \Big\} - 2 - K_2' \\
&\geq \frac{1}{2}(d_U(p_1, p_2) - 2) - 2 - K_2' = \frac{1}{2} d_U(p_1, p_2) - 3 - K_2'.
\end{aligned}$$

□

3.3.2 Proof of Theorem E

Consider an oriented hyperbolic surface B of type (g, n), which has n cusps. Let U_i be the cusp region of the i-th cusp, which is of area 2 and bounded by a horocycle of length 2, for $i = 1, \ldots, n$. The complement is a compact hyperbolic surface with boundary, denoted by $B_{cp} \subset B$.

The proof of Theorem E - (i) is made up of two lemmata. The first lemma claims that the holomorphic map restricted to a cusp region U_i is a quasi-isometric embedding whose parameters depend not only on (g, n), h and $\mathrm{sys}(B)$ but also on $\mathrm{sys}(F(b))$ for an arbitrary point $b \in B_{cp}$. The second lemma claims that $\mathrm{sys}(F(b))$ is bounded uniformly for $b \in B_{cp}$. Theorem E - (ii) is a consequence of Theorem E - (i) due to the fact that $\mathrm{diam}(\overline{B_{cp}})$ has an upper bound based on $\mathrm{sys}(B)$.

Lemma 3.3.8. *Given $\epsilon > 0$, there exists a constant $K_3 = K_3(g, n, h, \epsilon)$ that depends only on (g, n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B, t), \mathrm{Mod}_h)$ such that a peripheral monodromy of the i-th cusp is of infinite order, for some $i = 1, \ldots, n$. Suppose that* $\mathrm{sys}(B) \geq \epsilon$ *and* $\mathrm{sys}(F(b)) \geq \epsilon$ *for some $b \in B_{cp}$. Then, we have*

$$\frac{1}{2} d_B(p_1, p_2) \geq d_{\mathcal{M}}(F(p_1), F(p_2)) \geq \frac{1}{2} d_B(p_1, p_2) - K_3$$

for each pair of points $(p_1, p_2) \in U_i \times U_i$.

Proof. Since F is holomorphic, it is automatically distance-decreasing for $(1/2)d_B$ on B and $d_{\mathcal{M}}$ on $\mathcal{M}_h$. Now regard the hyperbolic surface B as the union of the compact region B_{cp} and n more disjoint cusp regions $U_1, \ldots, U_n$ each bounded by a horocycle of length 2. Select an arbitrary boundary point $p_{\max,i} \in \partial U_i$ and let $\mathrm{diam}\,(B_{cp})$ be the diameter of B_{cp}.

A peripheral monodromy of the i-th cusp, denoted by ϕ, is reducible, of infinite order and has no pseudo-Anosov reduced component. Therefore, some power ϕ^μ is identical on each component, where μ is bounded above by a constant determined by h. Hence, ϕ^μ is a multi-twist.

Suppose that $\mathrm{sys}(F(b)) \geq \epsilon$ for some $b \in B_{cp}$. Let (p_1, p_2) be a pair of points in U_i. By Theorem 3.3.7, the difference between $d_{\mathcal{M}}(F(p_1), F(p_2))$ and $(1/2)d_B(p_1, p_2)$ is bounded by a constant depending only on h, μ and a lower bound of $\mathrm{sys}(F(p_{\max,i}))$. By Mumford's compactness, the diameter $\operatorname{diam}(B_{cp})$ is bounded by a constant determined by (g,n) and ϵ. Wolpert's Lemma then shows a mutual dependence between $\mathrm{sys}(F(b))$ and $\mathrm{sys}(F(p_{\max,i}))$. Hence, the unique parameter used in the desired inequality depends only on (g,n), h and ϵ. □

Lemma 3.3.9. *Given $\epsilon > 0$, there exists a constant $K_4 = K_4(g,n,h,\epsilon)$ that depends only on (g,n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g,n) such that $\mathrm{sys}(B) \geq \epsilon$ and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Then, for any $p \in B_{cp}$, we have*

$$\mathrm{sys}(F(p)) \geq K_4.$$

Proof. One can derive this from Theorem 3.2.1 and Wolpert's Lemma. □

3.3.3 Proof of Theorem F

It remains to investigate a pair of points in distinct cusp regions. Theorem F comes from the following lemma.

Lemma 3.3.10. *Given $\epsilon > 0$, there exists a constant $K_5 = K_5(g,n,h,\epsilon)$ that depends only on (g,n), h, ϵ and satisfies the following statement. Let B be an oriented hyperbolic surface of type (g,n) such that $\mathrm{sys}(B) \geq \epsilon$ and $D \subset \mathbb{H}^2$ be a fundamental convex polygon of B with exactly n ideal points. Let $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map with a monodromy homomorphism $F_* \in \mathrm{Hom}(\pi_1(B,t), \mathrm{Mod}_h)$. For some $i \neq j$, $i = 1, \ldots, n$ and $j = 1, \ldots, n$, if peripheral monodromies of the i-th and the j-th cusps are of infinite order and they are not disjointed along some geodesic segment $\kappa_{i,j} \subset B$ having a lift $\widetilde{\kappa_{i,j}} \subset D$, then we have*

$$\frac{1}{2} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_1}), \widetilde{F}(\widetilde{p_2})) \geq \frac{1}{4} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) - K_5 - l_B(\kappa_{i,j})$$

for each pair of points $(p_1, p_2) \in U_i \times U_j$, where $\widetilde{p_1} \in D$ is a lift of p_1 and $\widetilde{p_2} \in D$ is a lift of p_2.

Proof. Without loss of generality, we assume that $i = 1$ and $j = 2$. Suppose that $p_1 \in H_{\epsilon_1} \subset U_1$ where H_{ϵ_1} is the horocycle of length $0 < \epsilon_1 \leq 2$ within the cusp region U_1. Suppose that $p_2 \in H_{\epsilon_2} \subset U_2$ where H_{ϵ_2} is the horocycle of length $0 < \epsilon_2 \leq 2$ within the cusp region U_2.

Consider the peripheral monodromies ϕ_1 and ϕ_2 associated to κ. Therefore, some power $\phi_1^{\mu_1}$ is a multi-twist along a multi-curve $\boldsymbol{\alpha}_1$ and some power $\phi_2^{\mu_2}$ is a multi-twist along a multi-curve $\boldsymbol{\alpha}_2$, where both μ_1 and μ_2 are bounded above by a constant determined by h. There exist $\alpha_1 \in \boldsymbol{\alpha}_1$ and $\alpha_2 \in \boldsymbol{\alpha}_2$ such that $\iota(\alpha_1, \alpha_2) \geq 1$.

Take $q_1 = F(p_1)$, $q_2 = F(p_2) \in \mathcal{M}_h$ and set $\widetilde{q_1} = \widetilde{F}(\widetilde{p_1})$, $\widetilde{q_2} = \widetilde{F}(\widetilde{p_2}) \in \mathcal{T}_h$. By Proposition 3.3.6, since $d_{\mathcal{T}}(\widetilde{q_1}, \phi_1^{\mu_1} \cdot \widetilde{q_1}) \leq \mu_1 \cdot \epsilon_1 \leq 2\mu_1$ and $d_{\mathcal{T}}(\widetilde{q_2}, \phi_2^{\mu_2} \cdot \widetilde{q_2}) \leq \mu_2 \cdot \epsilon_2 \leq 2\mu_2$ we have

$$L_{\alpha_1}(\widetilde{q_1}) \leq K_1(h,\mu_1) \cdot d_{\mathcal{T}}(\widetilde{q_1}, \phi_1^{\mu_1} \cdot \widetilde{q_1}) \leq K_1(h,\mu_1) \cdot \mu_1 \cdot \epsilon_1,$$
$$L_{\alpha_2}(\widetilde{q_2}) \leq K_1(h,\mu_2) \cdot d_{\mathcal{T}}(\widetilde{q_2}, \phi_2^{\mu_2} \cdot \widetilde{q_2}) \leq K_1(h,\mu_2) \cdot \mu_2 \cdot \epsilon_2.$$

Besides, since $\iota(\alpha_1, \alpha_2) \geq 1$, we have

$$\sinh\Big(\frac{L_{\alpha_1}(\widetilde{q_1})}{2}\Big) \sinh\Big(\frac{L_{\alpha_2}(\widetilde{q_1})}{2}\Big) \geq 1 \quad \text{and} \quad \sinh\Big(\frac{L_{\alpha_1}(\widetilde{q_2})}{2}\Big) \sinh\Big(\frac{L_{\alpha_2}(\widetilde{q_2})}{2}\Big) \geq 1$$

which implies that $L_{\alpha_1}(\widetilde{q_2}) \geq 2 \operatorname{arcsinh} \frac{1}{\sinh K_1(h,\mu_2)\cdot\mu_2}$ and $L_{\alpha_2}(\widetilde{q_1}) \geq 2 \operatorname{arcsinh} \frac{1}{\sinh K_1(h,\mu_1)\cdot\mu_1}$. Therefore, by Wolpert's Lemma, we have

$$\begin{aligned} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq 2 d_{\mathcal{T}}(\widetilde{F}(\widetilde{p_1}), \widetilde{F}(\widetilde{p_2})) = 2 d_{\mathcal{T}}(\widetilde{q_1}, \widetilde{q_2}) &\geq \frac{1}{2} \log \frac{L_{\alpha_1(\widetilde{q_2})}}{L_{\alpha_1(\widetilde{q_1})}} + \frac{1}{2} \log \frac{L_{\alpha_2(\widetilde{q_1})}}{L_{\alpha_2(\widetilde{q_2})}} \\ &\geq \frac{1}{2}\big(\log \frac{2}{\epsilon_1} + \log \frac{2}{\epsilon_2}\big) - K_{5,1,2}(h,\mu_1,\mu_2) - K_{5,2,1}(h,\mu_1,\mu_2) \end{aligned}$$

where

$$K_{5,1,2}(h,\mu_1,\mu_2) = \frac{1}{2} \log \frac{K_1(h,\mu_1) \cdot \mu_1}{\operatorname{arcsinh} \frac{1}{\sinh K_1(h,\mu_2)\cdot\mu_2}},$$
$$K_{5,2,1}(h,\mu_1,\mu_2) = \frac{1}{2} \log \frac{K_1(h,\mu_2) \cdot \mu_2}{\operatorname{arcsinh} \frac{1}{\sinh K_1(h,\mu_1)\cdot\mu_1}}.$$

Using triangle inequality in $(\mathbb{H}^2, d_{\mathbb{H}^2})$ and the fact that D is convex and bounded by geodesic segments, we conclude that

$$d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) \geq 2d_{\mathcal{T}}(\widetilde{q_1}, \widetilde{q_2}) \geq \frac{1}{2} d_{\mathbb{H}^2}(\widetilde{p_1}, \widetilde{p_2}) - \frac{1}{2}\operatorname{diam}(\widetilde{B_{cp}} \cap D) - K_{5,1,2}(h, \mu_1, \mu_2) - K_{5,2,1}(h, \mu_1, \mu_2)$$

where $\widetilde{B_{cp}} \subset \mathbb{H}^2$ is the lift of $B_{cp} \subset B$. $\square$

3.4 Examples and applications

This section is intended to provide several examples, remarks concerning and consequences of Theorem E and Theorem F. We focus on holomorphic curves in $\mathcal{M}_2$. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and τ, σ be closed curves on Σ_2 represented in Figure 3.3 and Figure 3.4.

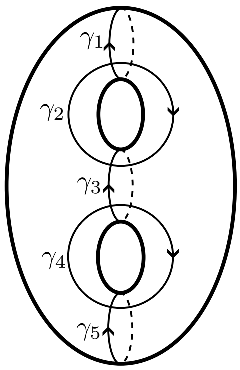


FIGURE 3.3 – Five closed curves on Σ_2 along which the Dehn twists generate Mod_2. We have chosen their orientations for later use.

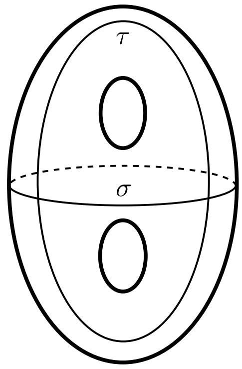


FIGURE 3.4 – Another pair of closed curves on Σ_2.

It is well-known (cf. [Bir75, Theorem 4.8] and [Aur03, Figure 1]) that Mod_2 is generated by the five Dehn twists $T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5}$ and admits the following presentation :

$$\mathrm{Mod}_2 = \left\langle T_{\gamma_1}, \ldots, T_{\gamma_5} \,\middle|\, \begin{array}{c} T_{\gamma_i} \circ T_{\gamma_j} = T_{\gamma_j} \circ T_{\gamma_i} \quad \text{if } |i-j| > 1\,; \\ T_{\gamma_i} \circ T_{\gamma_j} \circ T_{\gamma_i} = T_{\gamma_j} \circ T_{\gamma_i} \circ T_{\gamma_j} \quad \text{if } |i-j| = 1\,; \\ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_5}^2 \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2} \circ T_{\gamma_1} = I \text{ is central}\,; \\ I^2 = 1\,; \quad (T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_5})^6 = 1 \end{array} \right\rangle.$$

One can check that $I = (T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_4})^5$.

3.4.1 Quasi-isometrically but non-isometrically immersed curves

Theorem E provides a sufficient condition on the monodromy homomorphism for a holomorphic map $F : B \to \mathcal{M}_h$ to be a quasi-isometric immersion. On the other hand, the monodromy homomorphism of an isometric immersion $F : B \to \mathcal{M}_h$ is essentially purely pseudo-Anosov (see Definition 3.1.5 and Theorem 3.1.6). Therefore, we present a criterion for a holomorphic curve to be quasi-isometrically but not isometrically immersed.

Criterion 3.4.1. *Let B be an oriented hyperbolic surface of type (g, n) and $F : B \to \mathcal{M}_h$ be a non-constant holomorphic map. Suppose that (a) the monodromy homomorphism is not essentially purely pseudo-Anosov, (b) all peripheral monodromies are of infinite order. Then, the holomorphic curve $F(B) \subset \mathcal{M}_h$ is quasi-isometrically but not isometrically immersed.*

In the rest of this subsection, we construct a quasi-isometrically but not isometrically immersed holomorphic curve $F(B) \subset \mathcal{M}_2$ of type $(0, 6)$. In addition, there exists a desired fundamental polygon D as in Theorem F such that $\widetilde{F}\Big|_D$ is a quasi-isometric embedding.

Example 3.4.2. Set $B = \mathbb{C} \setminus \{-2, -1, 0, 1, 2\}$ and

$$C' = \left\{ \big([X_0 : X_1], [Y_0 : Y_1], b\big) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \times B \,\middle|\, \begin{aligned} X_0^6 Y_1^2 =& (X_1 + X_0 b)(X_1 - X_0 b)(X_1 + X_0) \\ & (X_1 + 2X_0)(X_1 - X_0)(X_1 - 2X_0) Y_0^2 \end{aligned} \right\}.$$

Let $\pi' : C' \to B$ be a holomorphic map with $\pi'([X_0 : X_1], [Y_0 : Y_1], t) = t$. Then, each fibre $\pi'^{-1}(b)$ is a double cover of $\mathbb{C}P^1$ via $\pi'^{-1}(b) \ni ([X_0 : X_1], [Y_0 : Y_1], t) \mapsto [X_0 : X_1] \in \mathbb{C}P^1$ with branch points

$$P_1 = [1 : b], P_2 = [1 : 2], P_3 = [1 : 1], P_4 = [1 : -1], P_5 = [1 : -2], P_6 = [1 : -b] \text{ and } \infty = [0 : 1].$$

Therefore, the resolution at $[0 : 1]$ for every $t \in B$ is a holomorphic family C/B of Riemann surfaces of genus 2, say $\pi : C \to B$, which is non-isotrivial.

Proposition 3.4.3. *The classifying map of the holomorphic family C/B in Example 3.4.2 is a quasi-isometric but not-isometric immersion. Moreover, the lift of the classifying map restricted to some fundamental polygon is a quasi-isometric embedding.*

Proof. The base B is a Riemann surface of type $(0, 6)$. To illustrate the monodromy homomorphism, we fix the base point $t := 3 \in B$ and investigate generic fibres at $b \in \Gamma \subset B$ where Γ is shown in Figure 3.5. In fact, the resolution of $\pi'^{-1}(t)$ is the union of two copies of a single-valued branch that are glued along the boundary, where the boundary consists of three connected components. Figure 3.6 shows a piecewise correspondence between the algebraic curve $\pi^{-1}(t)$ and the topological surface Σ_2.

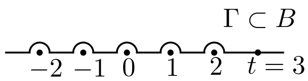


FIGURE 3.5 – The subset $\Gamma \subset B$ at each point of which the fibre is generic.

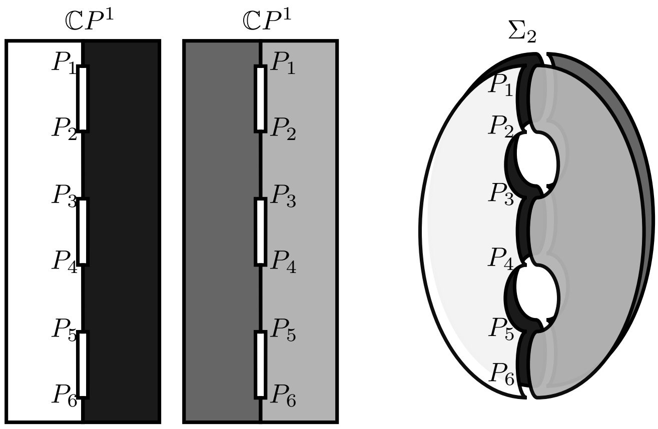


FIGURE 3.6 – A piecewise correspondence between two copies of the single-valued branch at $t = 3$ and Σ_2.

When $b \in \Gamma$ is approaching one of -2, -1, 0, 1 and 2, there exist several closed curves on the generic fibre each joining two distinct branch points and vanishing when b takes the limit. These closed curves are called *vanishing cycles*. Figure 3.7 tells us what the pair of branch points is for each vanishing cycle. The peripheral monodromy is not the product of Dehn twists along vanishing cycles, but the product of their squares. One may compare its action on a transverse arc to the standard picture of a Dehn twist and a squared Dehn twist (see Figure 3.8).

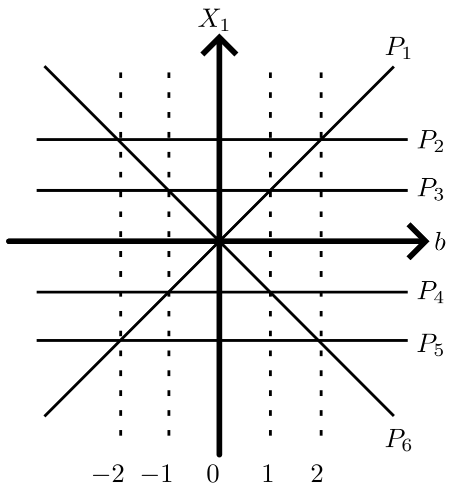


FIGURE 3.7 – Deformation of the branch points.

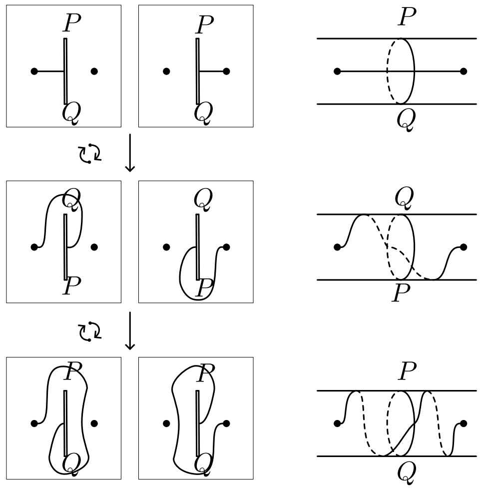


FIGURE 3.8 – A pair of branch points rotated clockwise and the squared Dehn twist.

To identify the vanishing cycles, we deform the algebraic curve $\pi^{-1}(b)$ along Γ and maintain the correspondence with Σ_2. This deformation and all vanishing cycles at -2, -1, 0, 1, 2 are depicted in Figure 3.9.

We need the following.

Lemma 3.4.4. *Let $\gamma_1, \gamma_2 \subset \Sigma_2$ be two closed curves on the (oriented) closed surface of genus 2 such that $\iota(\gamma_1, \gamma_2) = 1$. Consider a path, denoted by $\gamma_1 \triangle \gamma_2$, starting from some $p \in \gamma_1 \setminus \gamma_2$, moving along γ_1 to the intersection, turning right, moving along γ_2 back to the intersection, turning left and moving along γ_1 back to p. Then, we have*

$$T_{\gamma_1 \triangle \gamma_2} = T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2}.$$

Proof of Lemma 3.4.4. This comes from the fact that $\gamma_1 \triangle \gamma_2$ is homotopic to $T_{\gamma_2}^{-1}(\gamma_1)$. $\square$

We return to the proof of Proposition 3.4.3. The monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_2$ is expressed as a sextuple, denoted by $(\phi_\infty, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2)$, where ϕ_∞, ϕ_{-2}, ϕ_{-1}, ϕ_0, ϕ_1 and ϕ_2 are peripheral monodromies at ∞, -2, -1, 0, 1 and 2. Using $\phi_2^{1/2}$, $\phi_1^{1/2}$, $\phi_0^{1/2}$, $\phi_{-1}^{1/2}$ and $\phi_{-2}^{1/2}$ to denote half peripheral monodromies at 2, 1, 0, -1 and -2 respectively, so that

$$\begin{aligned}
\phi_2^{1/2} =& T_{\gamma_1} \circ T_{\gamma_5},\\
\phi_1^{1/2} =& T_{\gamma_1 \triangle \gamma_2} \circ T_{\gamma_5 \triangle \gamma_4} = T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4},\\
\phi_0^{1/2} =& T_{(\gamma_1 \triangle \gamma_2) \triangle (\gamma_5 \triangle \gamma_4 \triangle \gamma_3)}\\
=& T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5}^{-1} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3},\\
\phi_{-1}^{1/2} =& T_{\gamma_1 \triangle \gamma_2 \triangle \gamma_3} \circ T_{\gamma_5 \triangle \gamma_4 \triangle \gamma_3} = T_{\gamma_3}^{-1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3},\\
\phi_{-2}^{1/2} =& T_{\gamma_1 \triangle \gamma_2 \triangle \gamma_3 \triangle \gamma_4} \circ T_{\gamma_4 \triangle \gamma_3 \triangle \gamma_2 \triangle \gamma_1}\\
=& T_{\gamma_4}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2},
\end{aligned}$$

we observe that

$$T_\tau^2 \circ (\phi_{-2}^{1/2})^2 \circ (\phi_{-1}^{1/2})^2 \circ (\phi_0^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ (\phi_2^{1/2})^2 = 1$$

and

$$(\phi_2^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ (\phi_2^{1/2})^2 \circ (\phi_1^{1/2})^2 \circ T_{\gamma_2}^4 \circ T_{\gamma_4}^4 \circ T_\sigma^{-2} = 1$$

where τ and σ are given in Figure 3.4. Therefore, the peripheral monodromy at ∞ is again a multi-twist and there exists an essential closed curve on which the monodromy is a multi-twist. Hence, the holomorphic family C/B induces a quasi-isometrically but not isometrically immersed holomorphic curve.

The global monodromy $(\phi_\infty, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2)$ is a tuple in Mod_2 whose components are of infinite order and pairwise intersecting. Hence, the lift of the classifying map restricted to some fundamental polygon, say $\widetilde{F}\Big|_D : D \to \mathcal{T}_2$, is a quasi-isometric embedding. $\square$

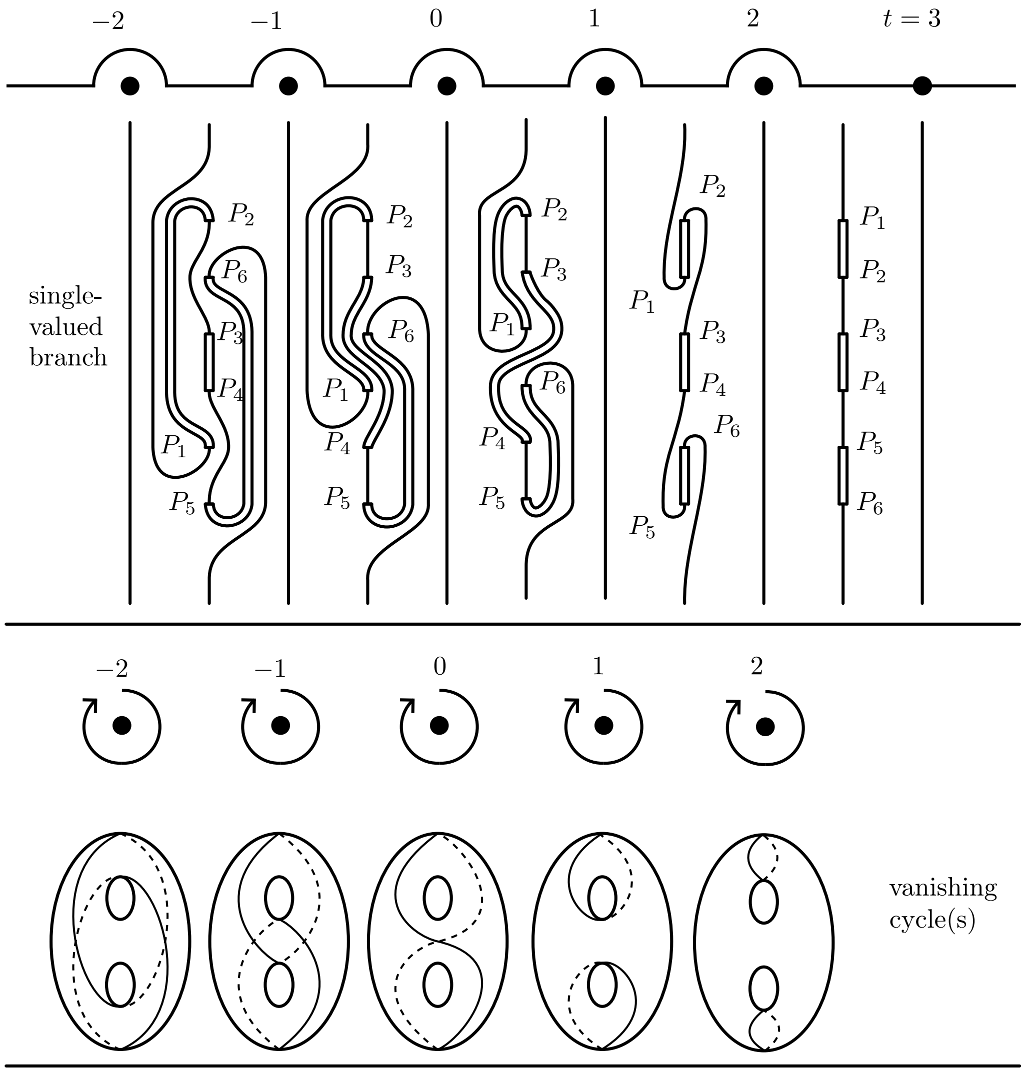


FIGURE 3.9 – Deformation of the generic fibre along Γ is illustrated by the deformation of the single-valued branch, due to the correspondence between two copies of single-valued branch and Σ_2. One can further point out the vanishing cycle(s) at 2, 1, 0, −1 and −2 respectively.

3.4.2 Non quasi-isometrically embedded cusp regions

In this subsection, we provide a holomorphic curve of type $(0, 8)$ in $\mathcal{M}_2$ for which a cusp region is not quasi-isometrically embedded. In fact, the corresponding peripheral monodromy is of finite order and therefore this holomorphic curve does not satisfy the hypothesis of Theorem E - (i).

Example 3.4.5. Set $B = \mathbb{C} \setminus \{-1, -1/2, -1/3, 0, 1/3, 1/2, 1\}$ and

$$C = \left\{ \big([X_0 : X_1], [Y_0 : Y_1], b\big) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \times B \;\middle|\; \begin{aligned} X_0^5 Y_1^2 = &(X_1 - (3b+2)X_0)(X_1 - bX_0) \\ &(X_1 + (3b-2)X_0)(X_1 + bX_0) \\ &(X_1 - X_0)Y_0^2 \end{aligned} \right\}.$$

Let $\pi : C \to B$ be a holomorphic map with $\pi([X_0 : X_1], [Y_0 : Y_1], t) = t$. Then, each fibre $\pi^{-1}(b)$ is a double cover of $\mathbb{C}P^1$ via $\pi^{-1}(b) \ni ([X_0 : X_1], [Y_0 : Y_1], t) \mapsto [X_0 : X_1] \in \mathbb{C}P^1$ with branch points

$$P_1 = [1 : 3b+2], P_2 = [1 : b], P_3 = [1 : 1], P_4 = [1 : -b], P_5 = [1 : -3b+2] \text{ and } \infty = [0 : 1].$$

Therefore, C/B is a holomorphic family of Riemann surfaces of genus 2, say $\pi : C \to B$, which is non-isotrivial.

Proposition 3.4.6. *The classifying map $F : B \to \mathcal{M}_2$ of the holomorphic family C/B in Example 3.4.5 satisfies the following properties :*

- *Peripheral monodromies at ∞ are of order* 2.
- *The restriction of F to the cusp region at ∞ lies in a thick part of $\mathcal{M}_2$.*

Proof. The base B is a Riemann surface of type $(0, 8)$. To illustrate the monodromy homomorphism, we fix the base point $t := 2 \in B$ and investigate generic fibres at $b \in \Gamma \subset B$ where Γ is given in Figure 3.10.

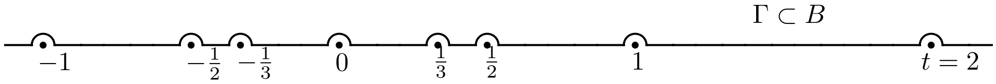


FIGURE 3.10 – The subset $\Gamma \subset B$ of generic positions we are looking at.

Figure 3.11 shows a piecewise correspondence between $C_t := \pi^{-1}(t)$ and the topological surface Σ_2. Therefore, the monodromy homomorphism $F_* : \pi_1(B, t) \to \mathrm{Mod}_2$ is expressed as an octuple, denoted by $(\phi_\infty, \phi_{-1}, \phi_{-1/2}, \phi_{-1/3}, \phi_0, \phi_{1/3}, \phi_{1/2}, \phi_1)$, where ϕ_∞, ϕ_{-1}, $\phi_{-1/2}$, $\phi_{-1/3}$, ϕ_0, $\phi_{1/3}$, $\phi_{1/2}$ and ϕ_1 are peripheral monodromies at ∞, -1, $-1/2$, $-1/3$, 0, 1/3, 1/2 and 1.

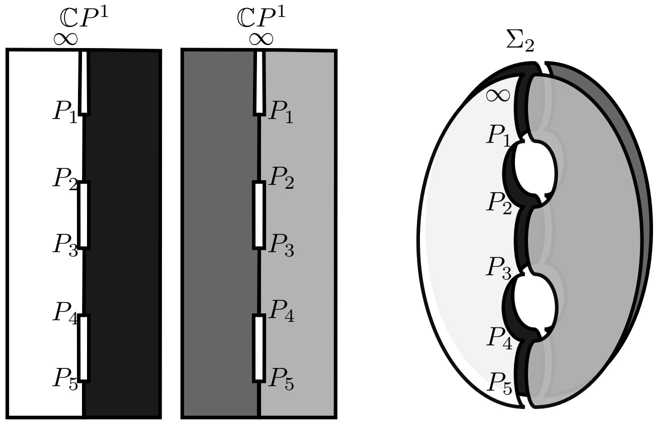


FIGURE 3.11 – A piecewise correspondence between C_t and Σ_2.

The half peripheral monodromies are given by the following.

$$
\begin{aligned}
\phi_1^{1/2} &= T_{\gamma_3} \circ T_{\gamma_5}, \\
\phi_{1/2}^{1/2} &= T_{\gamma_3 \triangle (\gamma_5 \triangle \gamma_4)} = T_{\gamma_4}^{-1} \circ T_{\gamma_5}^{-1} \circ T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4}, \\
\phi_{1/3}^{1/3} &= T_{\gamma_5 \triangle \gamma_4} = T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4}, \\
\phi_0^{1/2} &= T_{\gamma_3 \triangle \gamma_4} \circ T_{(\gamma_5 \triangle \gamma_4) \triangle (\gamma_3 \triangle \gamma_2)} \\
&= T_{\gamma_4}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3}^{-1} \circ T_{\gamma_2} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_5} \circ T_{\gamma_4} \circ T_{\gamma_2}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_2}, \\
\phi_{-1/3}^{1/2} &= T_{\gamma_3 \triangle \gamma_2} = T_{\gamma_2}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_2}, \\
\phi_{-1/2}^{1/2} &= T_{(\gamma_3 \triangle \gamma_4) \triangle \gamma_2)} = T_{\gamma_2}^{-1} \circ T_{\gamma_4}^{-1} \circ T_{\gamma_3} \circ T_{\gamma_4} \circ T_{\gamma_2}, \\
\phi_{-1}^{1/2} &= T_{\gamma_2} \circ T_{\gamma_4}.
\end{aligned}
$$

We observe that $(\phi_{-1}^{1/2})^2 \circ (\phi_{-1/2}^{1/2})^2 \circ (\phi_{-1/3}^{1/2})^2 \circ (\phi_0^{1/2})^2 \circ (\phi_{1/3}^{1/2})^2 \circ (\phi_{1/2}^{1/2})^2 \circ (\phi_1^{1/2})^2$ is of order 2.

Fix an orientation preserving diffeomorphism $f_t : \Sigma_2 \to C_t$ marking C_t and endow each $C_b := \pi^{-1}(b)$ with the marking f_b along Γ, for $b \in \Gamma$. We take a sufficiently large $N > 0$. To see that the restriction of F to the cusp region at ∞ lies in a thick part of $\mathcal{M}_2$, it suffices to show that $\mathrm{sys}(C_b)$ is bounded away from 0, for $b \in \mathbb{R}_{\geq N}$. From now on, we consider only generic positions $b \in \mathbb{R}_{\geq N}$ and generic fibres C_b at $b \in \mathbb{R}_{\geq N}$.

Half the hyperbolic distance on C_b is equal to the Kobayashi distance on C_b. Recall that the Kobayashi pseudo-norm on TC_b is defined by $\mathrm{Kob}_{C_b}(x, v) = \inf_\phi\{1/c\}$ for $x \in C_b$ and $v \in T_x C_b$, where the infimum is taken over all holomorphic maps $\phi : \Delta \to C_b$ satisfying $\phi(0) = x$ and $(d\phi)_0(\partial/\partial z) = c \cdot v$. In order to obtain a very coarse estimation of Kob_{C_b}, we make the following restrictions on ϕ : (i) $\phi(\Delta)$ lies in a single-valued branch, which is a subset of the affine chart $\mathbb{C}$;

(ii) $\phi : \Delta \to \mathbb{C}$ is the composition of a linear map and a translation, meaning that $\phi(\Delta) \subset \mathbb{C}$ is also a disc away from $P_1, \ldots, P_5$.

The branch points $P_1, \ldots, P_5$ are colinear. We observe that each ratio

$$r_{i,j,k,l}(b) \coloneqq \frac{d_{\mathbb{R}^2}(P_i, P_j)}{d_{\mathbb{R}^2}(P_k, P_l)}$$

of euclidean distances converges as $b \to \infty$, for $i \neq j$ and $k \neq l$. Set

$$r_{\min} \coloneqq (1/2) \min_{i,j,k,l} \{\lim_{b\to\infty} r_{i,j,k,l}(b)\} \quad \text{and} \quad r_{\max} \coloneqq (1/2) \max_{i,j,k,l} \{\lim_{b\to\infty} r_{i,j,k,l}(b)\}.$$

Therefore, for each $i \in \{2, 3, 4, 5\}$, there exists a closed curve β_i lying in a single-valued branch and homotopic to $f_b(\gamma_i)$ such that $l_{\mathbb{R}^2}(\beta_i) \leq 12 r_{\max} + \epsilon$ and $d_{\mathbb{R}^2}(x, P_j) \geq r_{\min} - \epsilon$ for all $x \in \beta_i$ and $j \in \{1, 2, 3, 4, 5\}$, where $\epsilon > 0$ is sufficiently small and determined by N (see Figure 3.12).

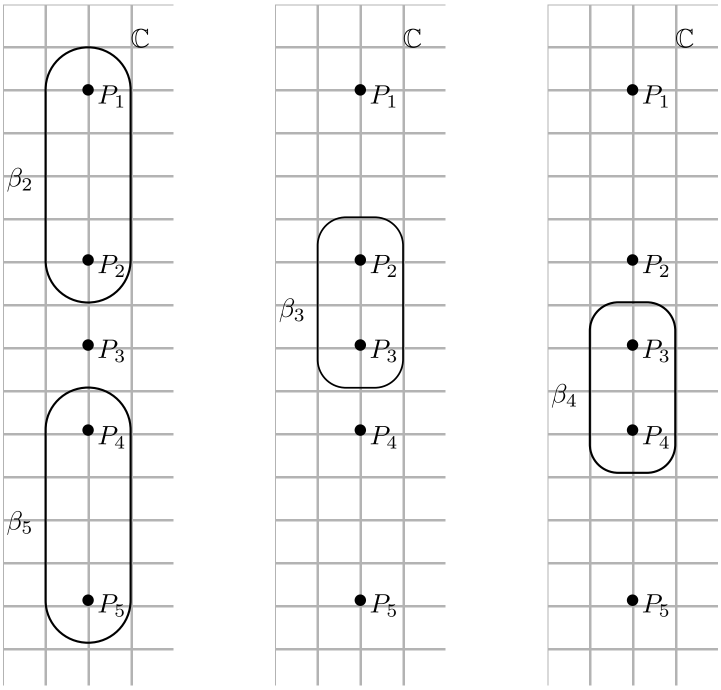


FIGURE 3.12 – Closed curves homotopic to $f_b(\gamma_2), \ldots, f_b(\gamma_5)$ with bounded Kobayashi lengths.

Each of β_2, β_3, β_4 and β_5 has the hyperbolic length

$$\begin{aligned} l_{C_b}(\beta_i) &= 2 \cdot \int_0^1 \mathrm{Kob}_{C_b}(\beta_i(t), \dot\beta_i(t)) dt \leq 2 \cdot \int_0^1 \frac{1}{\min_j \{d_{\mathbb{R}^2}(\beta_i(t), P_j)\}/|\dot\beta_i(t)|} dt \\ &\leq \frac{2}{r_{\min} - \epsilon} \int_0^1 |\dot\beta_i(t)| dt = \frac{2 l_{\mathbb{R}^2}(\beta_i)}{r_{\min} - \epsilon} \leq \frac{2(12 r_{\max} + \epsilon)}{r_{\min} - \epsilon}. \end{aligned}$$

Hence $L_{\gamma_2}(C_b)$, $L_{\gamma_3}(C_b)$, $L_{\gamma_4}(C_b)$ and $L_{\gamma_5}(C_b)$ are uniformly bounded from above. We conclude that $\mathrm{sys}(C_b)$ is bounded away from 0. □

3.4.3 Holomorphic genus-2 Lefschetz fibrations

Holomorphic genus-2 Lefschetz fibrations over a punctured sphere are quite well-understood by works of Siebert and Tian [ST05] as well as Chakiris [Cha83] and Smith [Smi99] (cf. also [Sal14]). In particular, there are only finitely many explicit possibilities for the global monodromies of genus-2 Lefschetz fibrations without reducible fibres (i.e. without separating vanishing cycles) up to Hurwitz moves. By convention, we use $\bullet$ to denote the concatenation of tuples : $(\phi_1, \ldots, \phi_k) \bullet (\psi_1, \ldots, \psi_l) = (\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_l)$. The power of a tuple corresponds to a repeated concatenation with itself. The symbol $\prod$ represents the concatenation of a family of tuples.

Proposition 3.4.7. *Given $n \geq 3$, let B be an oriented hyperbolic surface of type $(0, n)$ and $F : B \to \mathcal{M}_2$ be a holomorphic map with a global monodromy $(\phi_1, \ldots, \phi_n)$. Suppose each ϕ_i is the Dehn twist along a non-separating closed curve. Then the global monodromy has the following properties :*

(i) Using a finite sequence of Hurwitz moves, one can transform $(\phi_1, \ldots, \phi_n)$ into the concatenation of tuples $\mathcal{A}_1^p \bullet \mathcal{A}_2^q \bullet \mathcal{A}_3^r$ where p, q, r are non-negative integers and

$$\mathcal{A}_1 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5}, T_{\gamma_5}, T_{\gamma_4}, T_{\gamma_3}, T_{\gamma_2}, T_{\gamma_1})^2,$$
$$\mathcal{A}_2 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4})^5,\ \mathcal{A}_3 = (T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}, T_{\gamma_4}, T_{\gamma_5})^6.$$

(ii) Using a finite sequence of Hurwitz moves, one can transform $(\phi_1, \ldots, \ldots_n)$ into a tuple of Dehn twists along pairwise intersecting closed curves.

Proposition 3.4.7 - (i) is Theorem B in [Cha83] and the holomorphic map $F : B \to \mathcal{M}_2$ is the classifying map of a holomorphic genus-2 Lefschetz fibration without separating vanishing cycles. In this case, by Proposition 3.4.7 - (ii), all hypotheses of Theorem F hold for a specific fundamental polygon D and we have the following corollary.

Corollary 3.4.8. *Let $B = \Gamma\backslash\mathbb{H}^2$ be an oriented hyperbolic surface of type $(0, n)$, $n \geq 3$. Let $F : B \to \mathcal{M}_2$ be the classifying map of a holomorphic genus-2 Lefschetz fibration without separating vanishing cycles. Then, there exists a fundamental polygon D of B such that $\widetilde{F}\Big|_D : (D, (1/2)d_{\mathbb{H}^2}) \to (\mathcal{T}_2, d_{\mathcal{T}})$ is a $(2, K + \mathrm{diam}(D))$-quasi-isometric embedding, where $K = K(0, n, 2, \mathrm{sys}(B))$ as in Theorem F.*

For positive integer l, recall that Hurwitz moves acting on a l-tuple $(\phi_1, \ldots, \phi_k)$ are given by

$$(\ldots, \phi_i \circ \phi_{i+1} \circ \phi_i^{-1}, \phi_i, \ldots) \xleftarrow{L_i} (\ldots, \phi_i, \phi_{i+1}, \ldots) \xrightarrow{R_i} (\ldots, \phi_{i+1}, \phi_{i+1}^{-1} \circ \phi_i \circ \phi_{i+1}, \ldots).$$

We consider a closed curve $\delta \subset \Sigma_2$ and use $\overrightarrow{\delta}$ to denote an orientation of δ. The *algebraic intersection number* of two oriented closed curves $\overrightarrow{\delta_1}$ and $\overrightarrow{\delta_2}$, denoted by $\hat{\imath}(\overrightarrow{\delta_1}, \overrightarrow{\delta_2})$, is defined as the sum of the indices of the intersection points of $\overrightarrow{\delta_1}$ and $\overrightarrow{\delta_2}$, where an intersection point is of index $+1$ when the orientation of the intersection agrees with the orientation of $\Sigma_{g,n}$ and is -1 otherwise. Note that $\hat{\imath}(\overrightarrow{\delta_1}, \overrightarrow{\delta_2}) \neq 0$ only if $\iota(\delta_1, \delta_2) \neq 0$.

Let Ω_l be the set of l-tuples $(g_1, \ldots, g_l)$ where each g_i is a positive Dehn twist in Mod_2, ${}^\sharp\Omega_l$ be the set of l-tuples $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l})$ where each $\overrightarrow{\delta_i}$ is an orientation of some closed curve $\delta_i \subset \Sigma_2$. There is a natural map $\natural : {}^\sharp\Omega_l \to \Omega_l$ sending $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l})$ to $(T_{\delta_1}, \ldots, T_{\delta_l})$. We define the *matrix of algebraic intersections* $\hat{M} = \hat{M}(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) \in \mathbb{R}^{l \times l}$ on every element in ${}^\sharp\Omega_l$ by setting $\hat{M}_{i,j} = \hat{\imath}(\overrightarrow{\delta_i}, \overrightarrow{\delta_j})$.

The maps ${}^\sharp L_i$ and ${}^\sharp R_i$ on ${}^\sharp\Omega_l$ induced by the Hurwitz moves L_i and R_i are defined as follows.

$$(\ldots, T_{\delta_i}(\overrightarrow{\delta_{i+1}}), \overrightarrow{\delta_i}, \ldots) \xleftarrow{{}^\sharp L_i} (\ldots, \overrightarrow{\delta_i}, \overrightarrow{\delta_{i+1}}, \ldots) \xrightarrow{{}^\sharp R_i} (\ldots, \overrightarrow{\delta_{i+1}}, T^{-1}_{\delta_{i+1}}(\overrightarrow{\delta_i}), \ldots).$$

We also have the maps ${}^\flat L_i$ and ${}^\flat R_i$ on $\mathbb{R}^{l \times l}$ defined by

$$\hat{M}' = (m'_{j,k}) \xleftarrow{{}^\flat L_i} \hat{M} = (m_{j,k}) \xrightarrow{{}^\flat R_i} \hat{M}'' = (m''_{j,k})$$

such that $m'_{i,i} = m'_{i+1,i+1} = m''_{i,i} = m''_{i+1,i+1} = 0$, $m'_{i,i+1} = m_{i+1,i} = m''_{i,i+1}$, $m'_{i+1,i} = m_{i,i+1} = m''_{i+1,i}$ and, for $j, k \notin \{i, i+1\}$, that $m'_{j,k} = m_{j,k} = m''_{j,k}$,

$$m'_{i,k} = m_{i+1,k} + m_{i+1,i}m_{i,k},\ m'_{i+1,k} = m_{i,k}\ , m'_{j,i} = m_{j,i+1} - m_{i,i+1}m_{j,i},\ m'_{j,i+1} = m_{j,i},$$
$$m''_{i,k} = m_{i+1,k},\ m''_{i+1,k} = m_{i,k} - m_{i,i+1}m_{i+1,k},\ m''_{j,i} = m_{j,i+1},\ m''_{j,i+1} = m_{j,i} + m_{i+1,i}m_{j,i+1}.$$

Suppose that q is a sequence of Hurwitz moves. We use ${}^\sharp q$ to denote the sequence of corresponding maps on ${}^\sharp\Omega_l$ and use ${}^\flat q$ to denote the sequence of corresponding maps on $\mathbb{R}^{l \times l}$.

Proposition 3.4.9. *Let $(g_1, \ldots, g_l) \in \Omega_l$ be a tuple of positive Dehn twists in Mod_2 where $g_i = T_{\delta_i}$ for $i = 1, \ldots, l$. Let q be a sequence of Hurwitz moves. Suppose that $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) \in {}^\sharp\Omega_l$ is a lift of $(g_1, \ldots, g_l)$ and $(\overrightarrow{\delta'_1}, \ldots, \overrightarrow{\delta'_l}) \in {}^\sharp\Omega_l$ is a lift of the resulting tuple $q \cdot (g_1, \ldots, g_l)$. Then*

$${}^\flat q \cdot \hat{M}(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_l}) = \hat{M}(\overrightarrow{\delta'_1}, \ldots, \overrightarrow{\delta'_l}).$$

Proof. It suffices to show that the following diagram is commutative.

$$\begin{tikzcd}
\Omega_l & \Omega_l \arrow[l, "L_i"'] \arrow[r, "R_i"] & \Omega_l \\
{}^\sharp\Omega_l \arrow[u, "\natural"] \arrow[d, "\hat{M}"'] & {}^\sharp\Omega_l \arrow[l, "{}^\sharp L_i"'] \arrow[r, "{}^\sharp R_i"] \arrow[u, "\natural"] \arrow[d, "\hat{M}"'] & {}^\sharp\Omega_l \arrow[u, "\natural"] \arrow[d, "\hat{M}"'] \\
\mathbb{R}^{l \times l} & \mathbb{R}^{l \times l} \arrow[l, "{}^\flat L_i"'] \arrow[r, "{}^\flat R_i"] & \mathbb{R}^{l \times l}
\end{tikzcd}$$

On the one hand, we have $T_{T_{\delta_i}(\delta_{i+1})} = T_{\delta_i} \circ T_{\delta_{i+1}} \circ T_{\delta_i}^{-1}$ and $T_{T^{-1}_{\delta_{i+1}}(\delta_i)} = T^{-1}_{\delta_{i+1}} \circ T_{\delta_i} \circ T_{\delta_{i+1}}$ (see, e.g., [FM11, Fact 3.7]). On the other hand, the algebraic intersection is well-defined on homology classes and we have

$$[T_{\delta_i}(\overrightarrow{\delta_{i+1}})] = [\overrightarrow{\delta_{i+1}}] + \hat{\imath}(\overrightarrow{\delta_{i+1}}, \overrightarrow{\delta_i}) \cdot [\overrightarrow{\delta_i}], \quad [T^{-1}_{\delta_{i+1}}(\overrightarrow{\delta_i})] = [\overrightarrow{\delta_i}] - \hat{\imath}(\overrightarrow{\delta_i}, \overrightarrow{\delta_{i+1}}) \cdot [\overrightarrow{\delta_{i+1}}]$$

for $i = 1, \ldots, l-1$ (see, e.g., [FM11, Proposition 6.3]). □

We prove Proposition 3.4.7 - (ii) by starting with Lemma 3.4.10 with a computer-assisted proof.

Lemma 3.4.10. *For $i = 1, 2, 3$, there exists a finite sequence of Hurwitz moves q_i that satisfies the following statement. Suppose that $\mathcal{A}_i$ is transformed by q_i into a tuple of Dehn twists, denoted by $(T_{\delta_{i,1}}, \ldots, T_{\delta_{i,l_i}})$. Then the algebraic intersection between each two of $\delta_{i,1}, \ldots, \delta_{i,l_i}$ and γ_1 is non-zero.*

Proof. We consider the sequence of Hurwitz moves q_i below and show that the resulting l_i-tuple $(T_{\delta_{i,1}}, \ldots, T_{\delta_{i,l_i}}) = q_i \cdot \mathcal{A}_i$ satisfies the desired properties. Applying q_i to the (l_i+1)-tuple $\mathcal{A}_i \bullet \{T_{\gamma_1}\}$, we obtain a (l_i+1)-tuple of the form $(T_{\delta_{i,1}}, \ldots, T_{\delta_{i,l_i}}, T_{\gamma_1})$ since each component in q_i is neither L_{l_i} nor R_{l_i}. Suppose that $(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_{l_i+1}}) \in {}^\sharp\Omega_{l_i}$ is the lift of $\mathcal{A}_i \bullet \{T_{\gamma_1}\}$ where the orientations of $\overrightarrow{\gamma_1}, \ldots, \overrightarrow{\gamma_5}$ are shown in Figure 3.3. Consider the matrix of algebraic intersections $\hat{M} = \hat{M}(\overrightarrow{\delta_1}, \ldots, \overrightarrow{\delta_{l_i+1}})$ and apply ${}^\flat q_i$ to $\hat{M}$. By Proposition 3.4.9, it suffices to verify that ${}^\flat q_i \cdot \hat{M}$ has non-zero off-diagonal entries.

$$\begin{aligned}
q_1 = (&L_2, R_7, R_{10}, R_{11}, L_8, R_6, L_{16}, L_9, L_{10}, R_{11}, R_4, L_{16}, L_3, R_{18}, R_{19}, L_{10}, R_{13}, R_{17}, R_{18}, L_{17},\\
&L_4, R_{19}, R_{16}, L_{12}, L_{13}, R_{11}, R_{14}, R_3, R_{15}, L_5, L_6, R_5, L_1, L_{12}, R_{19}, R_4, L_6, R_9, L_8, R_4,\\
&L_7, L_8, R_5, R_6, R_{14}, R_5, R_4, L_{11}, R_{15}, R_{14}, R_4, R_{15}, R_9, L_{13}, L_{10}, R_{11}, L_7, L_{12}, R_4, R_{18},\\
&R_3, L_{10}, R_{16}, L_{15}, R_{13}, R_{12}, R_{12}, L_2, R_{16}, R_9, L_{11}, L_4, R_{19}, R_{14}, L_{15}, R_2, R_6, L_1, R_2, L_8,\\
&R_{12}, R_{16}, L_{17}, L_{18}),\\
q_2 = (&R_{14}, R_3, L_5, R_4, R_3, L_{15}, L_5, L_{17}, L_{19}, L_{10}, R_{16}, L_6, R_{11}, R_9, L_{15}, L_{12}, L_{10}, L_1, L_6, L_{13},\\
&R_{14}, R_{13}, L_9, L_{15}, L_{17}, L_{10}, L_8, L_{14}, L_6, R_{18}, L_{19}, L_{18}, L_5, L_{13}, R_6, L_{12}, L_{15}, R_{10}, R_4, R_{13},\\
&L_8, L_{18}, R_9, R_4, L_7, L_{16}, R_3, R_9, R_{11}, L_{16}, R_5, R_3, R_6, R_2),\\
q_3 = (&L_9, L_{17}, L_{11}, L_3, L_{15}, L_4, R_{25}, R_{26}, R_{25}, L_{16}, R_4, L_{20}, L_{19}, L_{15}, L_{16}, L_{16}, L_8, R_{24}, L_1, L_{27},\\
&L_7, R_9, L_{10}, L_9, L_{12}, L_{18}, R_{17}, L_4, R_5, R_{16}, R_{23}, L_{26}, L_{25}, R_8, R_1, R_7, R_7, R_3, L_{14}, R_2,\\
&R_9, R_{18}, L_{20}, R_{23}, R_5, R_4, R_{22}, R_{13}, L_{15}, L_4, L_{20}, L_2, L_{24}, R_6, R_6, L_1, L_{21}, L_5, R_{23}, L_{24},\\
&L_{22}, L_{23}, L_{19}, L_{17}, L_6, L_9, L_{15}, L_{23}, L_{29}, R_{21}, L_{29}, L_7, R_{24}, R_{15}, R_{22}, L_{29}, R_7, L_{21}, L_4, L_{22},\\
&R_6, R_{25}, R_{20}, L_2, L_{24}, L_{22}, R_6, L_{19}, R_9, R_1, L_{26}, R_5, L_4, R_{12}, L_{10}, R_7, R_{18}, L_{10}, R_{27}, L_5,\\
&R_{11}, L_3, L_6, R_{23}, R_8, L_9, R_4, R_2, R_1, R_{27}, R_{28}, L_{29}, L_9, L_{19}, L_{12}, L_{11}, R_1, R_{17}, R_{28}, L_{29},\\
&R_{27}, L_{28}, L_{27}, L_{16}, R_3, L_7, L_{29}, L_{28}, R_{15}, R_{17}, L_{13}, R_{18}, L_{26}, R_7, L_8, R_1, L_{21}, L_{20}, R_{16}, R_{25},\\
&L_8, L_2, R_{21}).
\end{aligned}$$

□

In the proof of Lemma 3.4.10 the matrix ${}^\flat q_i \cdot \hat{M}$ is hard to obtain manually but can be quickly solved by a computer. We implement these computations in Python and make our code available on GitHub : `https://github.com/AHdoc/HurwitzMoves_to_AlgIntersections`.

Proof of Theorem 3.4.7 - (ii). By Proposition 3.4.7 - (i) and Lemma 3.4.10, using a sequence of Hurwitz moves, one can always transform a global monodromy into the concatenation of sub-tuples

$$\mathcal{A}_u \bullet (T_{\delta_{1,1}}, \ldots, T_{\delta_{1,l_1}})^{p'-1} \bullet (T_{\delta_{2,1}}, \ldots, T_{\delta_{2,l_2}})^{q'-1} \bullet (T_{\delta_{3,1}}, \ldots, T_{\delta_{3,l_3}})^{r'-1}$$

with some $u \in \{1, 2, 3\}$ and non-negative integers p', q', r' such that $p' + q' + r' - 3 = p + q + r - 1$, $\hat{\imath}(\gamma_1, \delta_{i,j}) \neq 0$ for $i = 1, 2, 3$, $j = 1, \ldots, l_i$ and $\hat{\imath}(\delta_{i,j}, \delta_{i,k}) \neq 0$ for $i = 1, 2, 3$, $1 \leq j \neq k \leq l_i$. By Lemma 3.3.2, making N sufficiently large such that $N - 2 > \iota(\delta_{i,j}, \delta_{i',j'})$ for all $i = 1, 2, 3$, $i' = 1, 2, 3$, $1 \leq j \leq l_i$, $1 \leq j' \leq l_{i'}$, we have

$$\iota(T^N_{\gamma_1}(\delta_{i,j}), \delta_{i',j'}) \geq (N-2)\, \iota(\gamma_1, \delta_{i,j})\, \iota(\gamma_1, \delta_{i',j'}) - \iota(\delta_{i,j}, \delta_{i',j'}) \geq 1.$$

Since $T_{\delta_{i,1}} \cdots T_{\delta_{i,l_i}}$ is central for $i = 1, 2, 3$, one can further transform the global monodromy into

$$\begin{aligned}\mathcal{A}_u \bullet \prod_{j=1}^{p'} (T_{T_{\gamma_1}^{jN}(\delta_{1,1})}, \dots, T_{T_{\gamma_1}^{jN}(\delta_{1,l_1})}) \bullet \prod_{j=p'+1}^{p'+q'} (T_{T_{\gamma_1}^{jN}(\delta_{2,1})}, \dots, T_{T_{\gamma_1}^{jN}(\delta_{2,l_2})}) \\ \bullet \prod_{j=p'+q'+1}^{p'+q'+r'} (T_{T_{\gamma_1}^{jN}(\delta_{3,1})}, \dots, T_{T_{\gamma_1}^{jN}(\delta_{3,l_3})}).\end{aligned}$$

Applying Lemma 3.4.10 again, we replace $\mathcal{A}_u$ with $(T_{\delta_{u,1}}, \dots, T_{\delta_{u,l_u}})$, as desired. $\square$

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